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R. Rabah G.M. Sklyar P.Yu. Barkhayev

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Rabah Rabah[1], Grigory Sklyar[2] and Pavel Barkhayev[3]

[1] IRCCyN/École des Mines de Nantes,
4 rue Alfred Kastler, BP 44307,
F-44321 Nantes Cedex 3, France

[2] Institute of Mathematics, University of Szczecin,
70–451 Szczecin, Wielkopolska 15, Poland

[3] IRCCyN/École Centrale de Nantes
Permanent address: Department of Mathematics, Kharkov University,
4 Svobody sqr., Kharkov, 61077, Ukraine

E–mails:
rabah@irccyn.ec-nantes.fr,
sklar@univ.szczecin.pl,
pbarhaev@inbox.ru

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Abstract

We investigate the stability property of mixed retarded-neutral type systems. Considering an operator model of the system in Hilbert space we are interesting in the critical case when the spectrum of the operator belongs to the open left half-plane and there exists a sequence of eigenvalues with real parts approaching to zero. In this case the exponential stability is not possible and we are studying the strong asymptotic stability property. The present paper extends the results obtained in [R. Rabah, G.M. Sklyar, A. V. Rezounenko, Stability analysis of neutral type systems in Hilbert space. J. of Differential Equations, 214(2005), No. 2, 391–428] in which stability of systems of neutral type was studied using the existence of a Riesz basis of invariant finite-dimensional subspaces. However, for mixed retarded-neutral type systems such a basis may not exist for the whole state space. Though the main result on stability remains the same for mixed retarded-neutral type systems, the technic of its proof had to be changed and it involves a proof of resolvent boundedness on some invariant subspace. We show that the property of asymptotic stability is determinated not only by the spectrum of the system but also depends on geometrical characteristics of its main neutral term which in our situation may be singular. We also give an explicit example of two systems having the same spectrum in the open left half-plane, but one of them is asymptotically stable while the other one is unstable.
Keywords. Retarded-neutral type systems, exponential stability, strong stability, infinite dimensional systems.
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1 Introduction

The interest in considering delay differential equations and corresponding infinite-dimensional dynamical systems is caused by a huge amount of applied problem which can be described by these equations. The stability theory of such type of systems was studied intensively (see e.g. [3, 7, 9, 11, 12]). Number of results was obtained for retarded systems, however an analysis of neutral type systems is much more complicated and these systems are still studied not so deeply. In the case of neutral type systems one meets the notion of strong asymptotic (non-exponential) stability. This type of stability could happen if exponential stability is not possible. We note also that for retarded system exponential stability and strong asymptotic stability occur at the same time.

We consider systems given by the following functional differential equation:

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + \int_{-1}^{0} A_2(\theta)\dot{z}(t+\theta)\,\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)z(t+\theta)\,\mathrm{d}\theta, \tag{1.1}$$

where A_{-1} is a constant $n \times n$-matrix and A_2, A_3 are $n \times n$-matrices whose elements belongs to $L_2(-1, 0; \mathbb{C})$. The term $A_{-1}\dot{z}(t-1)$ is called principal.

In [18] an analysis of strong stability of systems (1.1) has been carried out in the case when the matrix A_{-1} of the principal neutral term is nonsingular. In this paper we want to avoid this restriction and to perform a stability analysis in case when $\det A_{-1}$ is allowed to be zero, i.e. in the case of systems of mixed retarded-neutral type.

We use the general theory of C_0-semigroups of linear bounded operators (see e.g. [28]). Following [18], we consider the operator model of the system introduced by Burns et al. [5] in product space:

$$\dot{x} = \mathcal{A}x, \qquad x(t) = \begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix}, \tag{1.2}$$

where $\mathcal{A}$ is the generator of a C_0-semigroup, which is defined by

$$\mathcal{A}x(t) = \mathcal{A}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \begin{pmatrix} \int_{-1}^{0} A_2(\theta)\dot{z}_t(\theta)\,\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)z_t(\theta)\,\mathrm{d}\theta \\ \mathrm{d}z_t(\theta)/\,\mathrm{d}\theta \end{pmatrix}, \tag{1.3}$$

with the domain

$$\mathcal{D}(\mathcal{A}) = \{(y, z(\cdot))^T : \ z \in H^1(-1,0;\mathbb{C}^n), y = z(0) - A_{-1}z(-1)\} \subset M_2, \tag{1.4}$$

where $M_2 \stackrel{def}{=} \mathbb{C}^n \times L_2(-1,0;\mathbb{C}^n)$ is the state space. In the case when $A_2(\theta) \equiv A_3(\theta) \equiv 0$ we use the notation $\widetilde{\mathcal{A}}$ for $\mathcal{A}$.

The dynamical system $(e^{t\mathcal{A}}, M_2)$ is said to be strongly asymptotically stable if for all $x \in M_2$: $\lim\limits_{t\to+\infty} e^{t\mathcal{A}}x = 0$.

In the paper [18] the following result on strong stability of neutral type systems of the form (1.1) was obtained.

Theorem (R. Rabah, G.M. Sklyar, A.V. Rezounenko, 2005). Let us assume that $\det A_{-1} \neq 0$. And we assume also that $\sigma(\mathcal{A}) \subset \{\lambda : \ \mathrm{Re}\lambda < 0\}$ and $\max\{|\mu| : \ \mu \in \sigma_1(A_{-1})\} = 1$. Let us put $\sigma_1 = \sigma(A_{-1}) \cap \{\mu : |\mu| = 1\}$. Then the following three mutually exclusive possibilities hold true:

1. σ_1 consists of simple eigenvalues only, i.e. an one-dimensional eigenspace corresponds to each eigenvalue and there are no root vectors. Then system (1.2) is asymptotically stable.

2. The matrix A_{-1} has a Jordan block, corresponding to an eigenvalue $\mu \in \sigma_1$. In this case system (1.2) is unstable.

3. There are no Jordan blocks, corresponding to eigenvalues from σ_1, but there exists an eigenvalue $\mu \in \sigma_1$ whose eigenspace is at least two dimensional. In this case system (1.2) can be either stable or unstable. Moreover, there exist two systems with the same spectrum, such that one of them is stable while the other one is unstable.

The present paper extends the last result on the case of mixed retarded-neutral type systems, i.e. on the case when $\det A_{-1}$ is allowed to be zero. Let us analyze the propositions of the theorem above in this case.

If the matrix A_{-1} has a Jordan block, corresponding to an eigenvalue $\mu \in \sigma_1$ (item 2), then the result remains the same: system (1.2) is unstable. The proof of this fact given in [18] does not use the assumption $\det A_{-1} \neq 0$ (this proof does not involve the Riesz basis technic).

The example given in [18] for illustrating the item 3 does not essentially need the assumption $\det A_{-1} \neq 0$. In this paper we give also an explicit method of construction of such examples (two systems with the same spectrum and with the same matrix A_{-1}, and one of these systems is stable while the other one is unstable) which does not use the requirement $\det A_{-1} \neq 0$. Thus, the dilemma of the item 3 holds also for mixed retarded-neutral type systems.

However, an extension of the result on stability (item 1 of the theorem) is not obvious at all. The reason of this is that the proof of stability in the case $\det A_{-1} \neq 0$ is essentially based on the results on existence of a Riesz basis of invariant finite-dimensional subspaces. However, for mixed retarded-neutral type systems such a basis may not exist for the whole state space M_2.

Let us describe this problem in more details. A motivation for involving Riesz basis notion in stability analysis could be found in [24, 25, 30]. It is noted in [30] that if the spectrum of the system is contained in a vertical strip and $\inf\{|\lambda - \lambda'|, \lambda, \lambda' \in \sigma(\mathcal{A}), \lambda \neq \lambda'\} > 0$, then the generalized eigenspaces form a Riesz basis [8]. On the other hand, in [23] the example was given which shows that the neutral type system (1.2) may not possess such a basis since its eigenvalues may be not separated.

It was shown in [18] that the state space M_2 possesses a Riesz basis of finite-dimensional subspaces which are $\mathcal{A}$-invariant (and $e^{t\mathcal{A}}$-invariant). The proof of stability is essentially based on this fundamental result which requires the condition $\det A_{-1} \neq 0$.

However, if $\det A_{-1} = 0$ then an infinite part of the spectrum of the operator $\mathcal{A}$ is not situated in vertical strips. Thus, we are not able to use the results on existence of the Riesz basis in the whole space and we need to involve another technic to investigate the stability property. Namely, we combine the Riesz basis technic with analysis of resolvent's boundedness on some invariant space.

We decompose the state space M_2 onto a direct sum of $\mathcal{A}$-invariant subspaces, one of which possesses the Riesz basis of finite-dimensional subspaces and, therefore, the Riesz basis technic may be applied in this subspace. To prove stability property on the second subspace we use the following criterion of exponential stability which could be found in [28] (Corollary 4.2.8 from Theorem 4.2.8, p.119) or in [14] (Theorem 3.35, p.139).

Theorem (on exponential stability). *Let $T(t)$ be a C_0-semigroup on a Hilbert space H with a generator A. Then $T(t)$ is exponentially stable if and only if the following conditions hold:*

1. $\{\lambda : \mathrm{Re}\lambda \geq 0\} \subset \rho(A)$;
2. $\|R(\lambda, A)\| \leq M$ *for all* $\{\lambda : \mathrm{Re}\lambda \geq 0\}$ *and for some constant* $M > 0$.

Combining two technics, we prove the following theorem:

Theorem 1.1 (on stability). *If $\sigma(\mathcal{A}) \subset \{\lambda :\ \mathrm{Re}\lambda < 0\}$ and $\sigma_1 = \sigma(A_{-1}) \cap \{\mu : |\mu| = 1\}$ consists only of simple eigenvalues, then system (1.1) is strongly asymptotically stable.*

We emphasize that we require no other conditions. In particular, we do not require the spectrum of the operator $\mathcal{A}^*$ to coincide with the spectrum of the operator $\mathcal{A}$.

We emphasize also that though the formulation of the main result on stability remains the same for mixed retarded-neutral type systems, the technic of its proof was essentially changed and it involves a proof of resolvent boundedness on some invariant subspace.

In [18] it was given an example of two systems having the same spectrum in the open left half-plane but one of them is asymptotically stable while the other one is unstable. That example was constructed mostly in infinite dimensional space M_2. In this paper we give an example of the same situation but in terms of systems of the form (1.1). The analysis of the spectrum being carried out in this example is essentially based on deep results on transcendental equations obtained in the paper of L. Pontryagin [17]. Such an example helps to understand better the situation that may occur in the case when there are no Jordan blocks, corresponding to eigenvalues in σ_1, but there exists an eigenvalue $\mu \in \sigma_1$ whose eigenspace is at least two-dimensional.

The paper is organized as follows. Sections 2–5 are devoted to a proof of Theorem 1.1 on stability of mixed neutral-retarded type systems. In Section 2 we construct a direct decomposition of the state space onto invariant subspaces and we give a proof of the main result on stability. In Section 3 we prove validity of the direct decomposition and in Section 4 we give a proof of resolvent boundedness on the invariant subspace. Section 5 is devoted to a proof of some auxiliary results which was used in proofs of the main results. In Section 6 we give an explicit example of two systems having the same spectrum in the open left half-plane but one of these systems is asymptotically stable while the other one is unstable.

2 Stability analysis

2.1 Notations and assumptions

Following the notation given in [18, 19], we introduce matrix-functions

$$\triangle(\lambda) = \triangle_{\mathcal{A}}(\lambda) = -\lambda I + \lambda e^{-\lambda} A_{-1} + \lambda \int_{-1}^{0} e^{\lambda s} A_2(s)\, \mathrm{d}s + \int_{-1}^{0} e^{\lambda s} A_3(s)\, \mathrm{d}s, \tag{2.5}$$

$$\triangle^*(\lambda) = \triangle_{\mathcal{A}^*}(\lambda) = -\lambda I + \lambda e^{-\lambda} A_{-1}^* + \lambda \int_{-1}^{0} e^{\lambda s} A_2^*(s)\, \mathrm{d}s + \int_{-1}^{0} e^{\lambda s} A_3^*(s)\, \mathrm{d}s, \tag{2.6}$$

which are in the relation $(\triangle(\lambda))^* = \triangle^*(\overline{\lambda})$ and the eigenvalues of the operators $\mathcal{A}$ and $\mathcal{A}^*$ are roots of the equations $\det \triangle(\lambda) = 0$ and $\det \triangle^*(\lambda) = 0$, respectively.

As it has been shown in [18, Proposition 1], the resolvent of the operator $\mathcal{A}$ has the following form:

$$R(\lambda, \mathcal{A}) \begin{pmatrix} z \\ \xi(\cdot) \end{pmatrix} \equiv \begin{pmatrix} e^{-\lambda} A_{-1} \int_{-1}^{0} e^{-\lambda s} \xi(s)\, \mathrm{d}s + (I - e^{-\lambda} A_{-1}) \triangle^{-1}(\lambda) D(z, \xi, \lambda) \\ \int_{0}^{\theta} e^{\lambda(\theta - s)} \xi(s)\, \mathrm{d}s + e^{\lambda\theta} \triangle^{-1}(\lambda) D(z, \xi, \lambda) \end{pmatrix},$$

where $\triangle(\lambda)$ is defined by (2.5) and $D(z, \xi, \lambda)$ is the following vector-function:

$$D(z, \xi) = D(z, \xi, \lambda) = z + \lambda e^{-\lambda} A_{-1} \int_{-1}^{0} e^{-\lambda\theta} \xi(\theta)\, \mathrm{d}\theta - \int_{-1}^{0} A_2(\theta) \xi(\theta)\, \mathrm{d}\theta$$

$$- \int_{-1}^{0} e^{\lambda\theta} [\lambda A_2(\theta) + A_3(\theta)] \left[\int_{0}^{\theta} e^{-\lambda s} \xi(s)\, \mathrm{d}s \right] \mathrm{d}\theta, \qquad z \in \mathbb{C}^n, \xi(\cdot) \in L_2(-1, 0; \mathbb{C}^n). \tag{2.7}$$

We denote by $\mu_1, \ldots, \mu_\ell$ the set of (distinct) eigenvalues of the matrix A_{-1} and by $p_1, \ldots, p_\ell$ their multiplicities; by $\widetilde{\lambda}_m^{(k)} = \ln|\mu_m| + \mathrm{i}(\arg \mu_m + 2\pi k)$, $m = \overline{1, \ell}$, $k \in \mathbb{Z}$ we denote eigenvalues of the operator $\widetilde{\mathcal{A}}$ and by $L_m^{(k)}(r^{(k)})$ – circles of the radius $r^{(k)} \leq \frac{1}{3}|\widetilde{\lambda}_m^{(k)} - \widetilde{\lambda}_i^{(j)}|$, $(m, k) \neq (i, j)$ centered at $\widetilde{\lambda}_m^{(k)}$.

We introduce a notation $\sigma_1 = \sigma(A_{-1}) \cap \{\mu : |\mu| = 1\}$ and, without loss of generality, we assume that $\{\mu_1, \ldots, \mu_{\ell_1}\} \subset \sigma_1$ and, thus, $p_1 = \ldots = p_{\ell_1} = 1$. We assume that the matrix A_{-1} is in Jordan form:

$$A_{-1} = \begin{pmatrix} \mu_1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \mu_{\ell_1} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & J_{\ell_1+1} & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & J_\ell \end{pmatrix}, \tag{2.8}$$

where J_m are Jordan blocks corresponding to eigenvalues $\mu_{\ell_1+1}, \ldots, \mu_\ell$.

We introduce the set of eigenvalues $\Lambda_1 = \Lambda_1(N)$:

$$\Lambda_1 = \Lambda_1(N) = \{\lambda_m^k : 1 \leq m \leq \ell_1, |k| \geq N\}, \tag{2.9}$$

and further we specify more precisely how large the number N should be.

According [19, Theorem 4], there exists N_1 such that for every index $m = \overline{1, \ell_1}$ and $k : |k| \geq N_1$ the total multiplicity of the roots of the equation $\det \triangle(\lambda) = 0$, contained in the circle $L_m^{(k)}(r^{(k)})$, equals $p_m = 1$, where radii $r^{(k)}$ satisfy relation $\sum\limits_{k\in\mathbb{Z}} (r^{(k)})^2 \leq \infty$. We denote these roots (eigenvalues of the operator $\mathcal{A}$) as λ_m^k, $1 \leq m \leq \ell_1$, $|k| \geq N_1$.

Without loss of generality, we assume that $r^{(k_1)} \geq r^{(k_2)}$ when $N_1 \leq k_1 < k_2$. Since for $m = \overline{1, \ell_1}$ all the circle $L_m^{(k)}(r^{(k)})$ are centered on the imaginary axis and their radii are approaching to zero, then let us chose $N \geq N_1$ big enough such that the strip $\{\lambda : \ -r^{(N)} < \mathrm{Re}\lambda < 0\}$ contains no eigenvalues of $\mathcal{A}$ except those belonging to circles $L_m^{(k)}(r^{(k)})$, $m = \overline{1, \ell_1}$, $|k| \geq N$.

2.2 Direct decomposition of the state space

Our aim is to divide the system onto exponentially stable part and strongly asymptotically stable part. To do this we construct a decomposition of the state space M_2 onto the direct sum of some $\mathcal{A}$-invariant subspaces. The method of construction is described below.

We split the spectrum of the operator $\mathcal{A}$:

$$\sigma(\mathcal{A}) = \Lambda_1 \cup \Lambda_0, \tag{2.10}$$

where Λ_1 consists of eigenvalues situated "not far away" from the imaginary axis, i.e., taking some small $\varepsilon > 0$, we put

$$\Lambda_1 = \sigma(\mathcal{A}) \cap \{\lambda : \ -\varepsilon < \mathrm{Re}\lambda < 0\} \tag{2.11}$$

and $\Lambda_0 = \sigma(\mathcal{A}) \backslash \Lambda_1$. We notice here that we take $\varepsilon > 0$ such small that Λ_1 consists only of eigenvalues "corresponding" to $\mu_i \in \sigma_1$ and the corresponding eigenvectors form a Riesz basis of the closure of their linear span. We give the precise way of construction the set Λ_1 in the proof of the main result.

We introduce an $\mathcal{A}$-invariant subspace M_2^1 consisting of eigenvectors of the operator $\mathcal{A}$:

$$M_2^1 = \ \mathrm{Cl} \ \mathrm{Lin}\{\varphi : \ (\mathcal{A} - \lambda I)\varphi = 0, \ \lambda \in \Lambda_1\}. \tag{2.12}$$

To construct another $\mathcal{A}$-invariant subspace (which we denote by M_2^0), we consider the closure of the linear span of eigenvectors of the operator $\mathcal{A}^*$ corresponding to eigenvalues from $\overline{\Lambda}_1$:

$$\widehat{M_2^1} = \ \mathrm{Cl} \ \mathrm{Lin}\{\psi : \ (\mathcal{A}^* - \overline{\lambda} I)\psi = 0, \ \lambda \in \Lambda_1\}. \tag{2.13}$$

Let us denote by M_2^0 the orthogonal complement to the subspace $\widehat{M_2^1}$ in M_2, i.e.:

$$M_2 = \widehat{M_2^1} \overset{\perp}{\oplus} M_2^0. \tag{2.14}$$

Since by the construction $\widehat{M_2^1}$ is an $\mathcal{A}^*$-invariant subspace, then M_2^0 is an $\mathcal{A}$-invariant subspace.

Finally, we consider the subspace $M_2^1 \oplus M_2^0$ and we want to prove that it coincides with the whole space M_2:

$$M_2 = M_2^1 \oplus M_2^0. \tag{2.15}$$

Our plan of proof of stability is the following: we prove the validity of the direct decomposition (2.15). Further, we prove that the resolvent of the operator $\mathcal{A}$ is uniformly bounded on the subspace M_2^0 and then, due to the theorem on exponential stability given above, we conclude that the restriction of the system (1.2) onto the $\mathcal{A}$-invariant subspace M_2^0 is exponentially stable. Proving also that the restriction of the system (1.2) onto the subspace M_2^1 is strongly stable, we obtain strong asymptotical stability of the system on the whole space M_2.

2.3 Proof of stability

Further we assume that the spectrum of the operator $\mathcal{A}$ belongs to the open left half-plane and also assume that the set $\sigma_1 = \sigma(A_{-1}) \cap \{\mu : |\mu| = 1\}$ consists only of simple eigenvalues (i.e. the multiplicity of these eigenvalues is equal to one).

The proof of Theorem 1.1 is based on the following theorems.

Theorem 2.1 (on direct decomposition). *There exists $N \in \mathbb{Z}$ such that the part of the spectrum $\Lambda_1 = \Lambda_1(N)$ given by (2.9) and the subspaces M_2^0, M_2^1, $\widehat{M_2^1}$, given by (2.12), (2.13) and (2.14), define the direct decomposition (2.15) of the space M_2, i.e.*

$$M_2 = M_2^1 \oplus M_2^0$$

and M_2^1, M_2^0 are $\mathcal{A}$-invariant subspaces.

Theorem 2.2 (on resolvent boundedness). *The restriction of the resolvent $R(\lambda, \mathcal{A})$ is uniformly bounded on the subspace M_2^0 for $\lambda : \mathrm{Re}\lambda \geq 0$, i.e. there exists $C > 0$ such that*

$$\|R(\lambda, \mathcal{A})x\| \leq C\|x\|, \quad x \in M_2^0. \tag{2.16}$$

Proofs of these two theorems are given below and now we prove the main result on stability.

Proof of Theorem 1.1. Let us show that for any $x \in M_2$: $\|e^{t\mathcal{A}}x\| \to 0$ when $t \to +\infty$. Due to Theorem 2.1 each $x \in M_2$ allows the following representation:

$$x = x_0 + x_1, \quad x_0 \in M_2^0,\ x_1 \in M_2^1.$$

It has been shown in [18, Theorem 15] that there exists $N \in \mathbb{Z}$ such that elements

$$\{\varphi_m^k :\ (\mathcal{A} - \lambda_m^k I)\varphi_m^k = 0,\ \lambda_m^k \in \Lambda_1 = \Lambda_1(N)\} \tag{2.17}$$

form a Riesz basis of the closure of their linear span. Thus, for any $x_1 \in M_2^1$ we have representations

$$x_1 = \sum_{\substack{1 \leq m \leq l_1 \\ |k| \geq N}} c_m^k \varphi_m^k, \qquad e^{t\mathcal{A}} x_1 = \sum_{\substack{1 \leq m \leq l_1 \\ |k| \geq N}} e^{\lambda_m^k t} c_m^k \varphi_m^k, \qquad \sum_{\substack{1 \leq m \leq l_1 \\ |k| \geq N}} |c_m^k|^2 < \infty.$$

Let us consider a norm $\|\cdot\|_1$ in which the Riesz basis (2.17) is orthogonal, then we have the following estimate:

$$\|e^{t\mathcal{A}} x_1\|_1 = \left(\sum_{\substack{1 \leq m \leq l_1 \\ |k| \geq N}} e^{2\mathrm{Re}\lambda_m^k t} \|c_m^k \varphi_m^k\|_1^2 \right)^{\frac{1}{2}} \leq \|x_1\|_1. \tag{2.18}$$

Since the series $\sum\limits_{\substack{1\le m\le l_1\\|k|\ge N}} c_m^k\varphi_m^k$ converges and since $\|\varphi_m^k\|_1$ is uniformly bounded then for any $\varepsilon>0$ there exists $N_1\ge N$ such that we have the estimate $\sum\limits_{\substack{1\le m\le l_1\\|k|\ge N_1}} \|c_m^k\varphi_m^k\|_1^2 \le \frac{\varepsilon^2}{8}$.

Moreover, since the set $\{(m,k)\ :\ 1\le m\le l_1,\ N\le |k|\le N_1\}$ is finite and since $\mathrm{Re}\lambda_m^k<0$, then there exists $t_0>0$ such that for any $t\ge t_0$ we have an estimate $\sum\limits_{\substack{1\le m\le l_1\\N\le|k|\le N_1}} e^{2\mathrm{Re}\lambda_m^k t}\|c_m^k\varphi_m^k\|_1^2 \le \frac{\varepsilon^2}{8}$.

Thus, we have

$$\sum_{\substack{1\le m\le l_1\\|k|\ge N}} e^{2\mathrm{Re}\lambda_m^k t}\|c_m^k\varphi_m^k\|_1^2 \le \sum_{\substack{1\le m\le l_1\\N\le|k|\le N_1}} e^{2\mathrm{Re}_m^k\lambda t}\|c_m^k\varphi_m^k\|_1^2 + \sum_{\substack{1\le m\le l_1\\|k|\ge N_1}} \|c_m^k\varphi_m^k\|_1^2 \le \frac{\varepsilon^2}{4}. \qquad (2.19)$$

Due to Theorem 2.2 the semigroup $e^{t\mathcal{A}}|_{M_2^0}$ is exponentially stable, i.e. by definition there exist some positive constants M, ω such that $\|e^{t\mathcal{A}}|_{M_2^0}\|\le Me^{-\omega t}$. Thus, for any $x_0\in M_2^0$ there exists $t_0>0$ such that for any $t\ge t_0$ we have an estimate

$$\|e^{t\mathcal{A}}x_0\|_1 \le Me^{-\omega t}\|x_0\|_1 \le \frac{\varepsilon}{2}. \qquad (2.20)$$

Finally, from the estimates (2.18), (2.19) and (2.20) we conclude that for any $x\in M_2$ and for any $\varepsilon>0$ there exists $t_0>0$ such that for any $t\ge t_0$ the following estimate holds:

$$\|e^{t\mathcal{A}}x\|_1 \le \|e^{t\mathcal{A}}x_0\|_1 + \|e^{t\mathcal{A}}x_1\|_1 \le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon. \qquad (2.21)$$

Therefore, $\lim\limits_{t\to+\infty}\|e^{t\mathcal{A}}x\|_1=0$ what implies that the system (1.2) is strongly asymptotically stable. $\square$

3 Proof of the direct decomposition

As it has been shown in [18] and [19] an eigenvector φ_m^k of the operator $\mathcal{A}$ corresponding to an eigenvalue λ_m^k is of the form

$$\varphi_m^k = \begin{pmatrix} (I-e^{-\lambda_m^k}A_{-1})x_m^k \\ e^{\lambda_m^k\theta}x_m^k \end{pmatrix}, \qquad (3.22)$$

where $x_m^k\in\mathrm{Ker}\triangle(\lambda_m^k)$; an eigenvector ψ_m^k of the operator $\mathcal{A}^*$ corresponding to an eigenvalue $\overline{\lambda_m^k}$ is of the form

$$\psi_m^k = \begin{pmatrix} y_m^k \\ \left[\overline{\lambda_m^k}e^{-\overline{\lambda_m^k}\theta} - A_2^*(\theta) + e^{-\overline{\lambda_m^k}\theta}\int_0^\theta e^{\overline{\lambda_m^k}s}A_3^*(s)\,\mathrm{d}s + \overline{\lambda_m^k}e^{-\overline{\lambda_m^k}\theta}\int_0^\theta e^{\overline{\lambda_m^k}s}A_2^*(s)\,\mathrm{d}s\right] y_m^k \end{pmatrix}, \qquad (3.23)$$

where $y_m^k\in\mathrm{Ker}\triangle^*(\overline{\lambda_m^k})$ and $\triangle^*(\lambda)$ is given by (2.6).

Lemma 3.1 *Let $\lambda_m^k\in\sigma(\mathcal{A})$, $\overline{\lambda_i^j}\in\sigma(\mathcal{A}^*)$ and φ_m^k, ψ_i^j are corresponding eigenvectors:*

$$(\mathcal{A}-\lambda_m^kI)\varphi_m^k=0, \qquad (\mathcal{A}^*-\overline{\lambda_i^j}I)\psi_i^j=0.$$

The scalar product $\langle \varphi_m^k, \psi_i^j \rangle_{M_2}$ *equals to the following value:*

$$\langle \varphi_m^k, \psi_i^j \rangle_{M_2} = \left\{ \begin{array}{cc} 0, & (m,k) \neq (i,j) \\ -\langle \triangle'(\lambda_m^k) x_m^k, y_m^k \rangle_{\mathbb{C}^n}, & (m,k) = (i,j) \end{array} \right. , \tag{3.24}$$

where x_m^k, y_m^k *are defined by (3.22) and (3.23).*

Lemma 3.2 *The sets of eigenvectors* $\{\varphi_m^k\}_{\substack{1\le m\le l_1\\ |k|\ge N}}$ *and* $\{\frac{1}{\lambda_m^k}\psi_m^k\}_{\substack{1\le m\le l_1\\ |k|\ge N}}$, *defined by (3.22) and (3.23) with* $\|x_m^k\|_{\mathbb{C}^n} = 1$, $\|y_m^k\|_{\mathbb{C}^n} = 1$, *are bounded, i.e. there exists a constant* $C > 0$ *such that*

$$\|\varphi_m^k\| \le C, \quad \|\frac{1}{\lambda_m^k}\psi_m^k\| \le C, \qquad 1 \le m \le l_1,\ |k| \ge N. \tag{3.25}$$

Lemma 3.3 *Let* $\lambda_m^k \in \Lambda_1$ *then there exists matrices*

$$P_{m,k} = \begin{pmatrix} 1 & -p_2 & \dots & -p_n \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \quad Q_{m,k} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ -q_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -q_n & 0 & \dots & 1 \end{pmatrix} \tag{3.26}$$

such that the matrix $\frac{1}{\lambda_m^k} R_m P_{m,k} \triangle(\lambda_m^k) R_m Q_{m,k}$ *has the following form:*

$$\widehat{\triangle}_{m,k}(\lambda_m^k) \stackrel{\text{def}}{=} \frac{1}{\lambda_m^k} P_{m,k} R_m \triangle(\lambda_m^k) R_m Q_{m,k} = \begin{pmatrix} 0 & 0 \ \dots \ 0 \\ 0 & \\ \vdots & S_{m,k} \\ 0 & \end{pmatrix}, \quad \det S_{m,k} \neq 0, \tag{3.27}$$

where

$$R_m = \begin{pmatrix} \widehat{R}_m & 0 \\ 0 & I \end{pmatrix}, \quad I = I_{n-m} \in \mathbb{C}^{n-m \times n-m}, \quad \widehat{R}_m = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathbb{C}^{m \times m}.$$

Moreover, for any $\varepsilon > 0$ *there exists* $N \in \mathbb{Z}$ *such that for any* $|k| \ge N$ *we have estimates on* $p_i = p_i(m,k)$, $q_i = q_i(m,k)$:

$$|p_i| \le \varepsilon,\ |q_i| \le \varepsilon, \quad i = \overline{2,n}. \tag{3.28}$$

Corollary 3.1 *In a neighborhood* $U(\lambda_m^k)$ *of each* $\lambda_m^k \in \Lambda_1$ *the matrix function* $\widehat{\triangle}(\lambda)$ *allows a representation*

$$\widehat{\triangle}_{m,k}(\lambda) = \begin{pmatrix} (\lambda - \lambda_m^k) r_{11}(\lambda) & (\lambda - \lambda_m^k) r_{12}(\lambda) \ \dots \ (\lambda - \lambda_m^k) r_{1n}(\lambda) \\ (\lambda - \lambda_m^k) r_{21}(\lambda) & \\ \vdots & M_{m,k}(\lambda) \\ (\lambda - \lambda_m^k) r_{n1}(\lambda) & \end{pmatrix}, \qquad \lambda \in U(\lambda_m^k), \tag{3.29}$$

where $r_{ij}(\lambda) = r_{ij}^{m,k}(\lambda)$ *are analytic functions. Moreover,*

$$r_{11}^{m,k}(\lambda_m^k) \neq 0, \qquad |r_{11}^{m,k}(\lambda_m^k)| \to 1,\ k \to \infty. \tag{3.30}$$

Remark 3.1 *Direct computations give us:*

$$P_{m,k}^{-1} = \begin{pmatrix} 1 & p_2 & \dots & p_n \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \quad Q_{m,k}^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ q_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_n & 0 & \dots & 1 \end{pmatrix}. \tag{3.31}$$

Lemma 3.4 *There exist constants* $0 < C_1 < C_2$, $N \in \mathbb{Z}$ *such that for any* $\lambda_m^k \in \Lambda_1 = \Lambda_1(N)$ *we have an estimate*

$$0 < C_1 \leq \left| \frac{1}{\lambda_m^k} \langle \triangle'(\lambda_m^k) x_m^k, y_m^k \rangle \right| \leq C_2, \qquad \lambda_m^k \in \Lambda_1. \tag{3.32}$$

Using the propositions given above we prove the theorem on direct decomposition.

Proof of Theorem 2.1. Let us prove that any element $x \in M_2$ allows the following representation:

$$x = x_0 + \sum_{\substack{1 \leq m \leq l_1 \\ |k| \geq N}} c_m^k \varphi_m^k, \quad x_0 \in M_2^0, \ \varphi_m^k \in M_2^1, \ \sum_{\substack{1 \leq m \leq l_1 \\ |k| \geq N}} |c_m^k|^2 < \infty. \tag{3.33}$$

It has been shown in [18, Theorem 15] that there exists $N \in \mathbb{Z}$ such that elements $\{\varphi_m^k : \ (\mathcal{A} - \lambda_m^k I)\varphi_m^k = 0, \ \lambda_m^k \in \Lambda_1 = \Lambda_1(N)\}$ form a Riesz basis of the closure of their linear span. Also the set $\{\frac{1}{\lambda_m^k}\psi_m^k : \ (\mathcal{A}^* - \overline{\lambda_m^k} I)\psi_m^k = 0, \ \lambda_m^k \in \Lambda_1 = \Lambda_1(N)\}$ form a Riesz basis of the closure of their linear span.

We choose $\|x_m^k\|_{\mathbb{C}^n} = 1$ and $\|y_m^k\|_{\mathbb{C}^n} = 1$ and, due to Lemma 3.2, we obtain that there exists $C > 0$ such that $\|\varphi_m^k\|_{M^2} \leq C$ and $\|\widehat{\psi}_m^k\|_{M^2} = \|\frac{1}{\lambda_m^k}\psi_m^k\|_{M^2} \leq C$ for all $\lambda_m^k \in \Lambda_1$.

Applying decomposition (2.14) to vectors φ_m^k we obtain

$$\varphi_m^k = \gamma_m^k + \sum_{\substack{1 \leq i \leq l_1 \\ |j| \geq N}} a_i^j \widehat{\psi}_i^j, \quad \gamma_m^k \in M_2^0.$$

Since, due to Lemma 3.1, $\langle \varphi_m^k, \widehat{\psi}_i^j \rangle = 0$ for $(m,k) \neq (i,j)$, then the last representation can be rewritten as follows:

$$\varphi_m^k = \gamma_m^k + a_m^k \widehat{\psi}_m^k, \quad \gamma_m^k \in M_2^0, \tag{3.34}$$

and, moreover, due to (3.24) we have

$$a_m^k = \frac{\langle \varphi_m^k, \widehat{\psi}_m^k \rangle_{M_2}}{\|\widehat{\psi}_m^k\|_{M_2}^2} = \frac{\frac{1}{\lambda_m^k}\langle \varphi_m^k, \psi_m^k \rangle_{M_2}}{\|\widehat{\psi}_m^k\|_{M_2}^2} = \frac{-\frac{1}{\lambda_m^k}\langle \triangle'(\lambda_m^k) x_m^k, y_m^k \rangle_{\mathbb{C}^n}}{\|\widehat{\psi}_m^k\|_{M_2}^2}. \tag{3.35}$$

We note also that

$$\|\gamma_m^k\| \leq \|\varphi_m^k\| + |a_m^k| \|\widehat{\psi}_m^k\| \leq C + \sqrt{|\frac{1}{\lambda_m^k}\langle \triangle'(\lambda_m^k) x_m^k, y_m^k \rangle|}. \tag{3.36}$$

Using decomposition (2.14) and the relation (3.34) we represent each vector $x \in M_2$ as follows:

$$x = \widehat{x}_0 + \sum_{\substack{1 \leq m \leq l_1 \\ |k| \geq N}} b_m^k \widehat{\psi}_m^k = \widehat{x}_0 - \sum_{\substack{1 \leq m \leq l_1 \\ |k| \geq N}} \frac{b_m^k}{a_m^k} \gamma_m^k + \sum_{\substack{1 \leq m \leq l_1 \\ |k| \geq N}} \frac{b_m^k}{a_m^k} \varphi_m^k = x_0 + \sum_{\substack{1 \leq m \leq l_1 \\ |k| \geq N}} c_m^k \varphi_m^k, \tag{3.37}$$

where $\widehat{x}_0 \in M_2^0$, $\sum\limits_{\substack{1\le m\le l_1\\ |k|\ge N}} |b_m^k|^2 < \infty$, $x_0 = \widehat{x}_0 - \sum\limits_{\substack{1\le m\le l_1\\ |k|\ge N}} \frac{b_m^k}{a_m^k}\gamma_m^k \in M_2^0$, $c_m^k = \frac{b_m^k}{a_m^k}$.

To prove the validity of the decomposition (3.37) it is enough to show that $\left|\frac{1}{a_m^k}\right| \le C_1$ and $\|\gamma_m^k\| \le C_2$. Taking into account (3.35) and (3.36), the last means to give an estimate

$$0 < C_1 \le \left| \frac{1}{\lambda_m^k} \langle \triangle'(\lambda_m^k) x_m^k, y_m^k \rangle \right| \le C_2, \qquad \lambda_m^k \in \Lambda_1, \tag{3.38}$$

which is proved by Lemma 3.4.

Therefore, the representation (3.37) holds for any $x \in M_2$ and this completes the proof of the theorem. $\square$

Now we prove the auxiliary lemmas.

Proof of Lemma 3.1. 1. Let us compute directly the scalar product $\langle \varphi_m^k, \psi_i^j \rangle_{M_2}$ using the representations (3.22) and (3.23):

$$\langle \varphi_m^k, \psi_i^j \rangle_{M_2} = \langle (I - e^{-\lambda_m^k} A_{-1}) x_m^k, y_i^j \rangle_{\mathbb{C}^n} + \int_{-1}^0 \langle e^{\lambda_m^k \theta} x_m^k, \overline{\lambda_i^j} e^{-\overline{\lambda_i^j}\theta} y_i^j \rangle_{\mathbb{C}^n} \,\mathrm{d}\theta - \int_{-1}^0 \langle e^{\lambda_m^k \theta} x_m^k, A_2^*(\theta) y_i^j \rangle_{\mathbb{C}^n} \,\mathrm{d}\theta$$

$$+ \int_{-1}^0 \langle e^{\lambda_m^k \theta} x_m^k, e^{-\overline{\lambda_i^j}\theta} \int_0^\theta e^{\overline{\lambda_i^j} s} A_3^*(s) \,\mathrm{d}s \cdot y_i^j \rangle_{\mathbb{C}^n} \,\mathrm{d}\theta + \int_{-1}^0 \langle e^{\lambda_m^k \theta} x_m^k, \overline{\lambda_i^j} e^{-\overline{\lambda_i^j}\theta} \int_0^\theta e^{\overline{\lambda_i^j} s} A_2^*(s) \,\mathrm{d}s \cdot y_i^j \rangle_{\mathbb{C}^n} \,\mathrm{d}\theta$$

$$= \langle (I - e^{-\lambda_m^k} A_{-1}) x_m^k, y_i^j \rangle + \langle \int_{-1}^0 \lambda_i^j e^{(\lambda_m^k - \lambda_i^j)\theta} \,\mathrm{d}\theta \cdot x_m^k, y_i^j \rangle - \langle \int_{-1}^0 e^{\lambda_m^k \theta} A_2(\theta) \,\mathrm{d}\theta \cdot x_m^k, y_i^j \rangle$$

$$+ \langle \int_{-1}^0 e^{(\lambda_m^k - \lambda_i^j)\theta} \int_0^\theta e^{\lambda_i^j s} A_3(s) \,\mathrm{d}s \,\mathrm{d}\theta \cdot x_m^k, y_i^j \rangle + \langle \lambda_i^j \int_{-1}^0 e^{(\lambda_m^k - \lambda_i^j)\theta} \int_0^\theta e^{\lambda_i^j s} A_2(s) \,\mathrm{d}s \,\mathrm{d}\theta \cdot x_m^k, y_i^j \rangle$$

$$= \langle \Gamma(\lambda_m^k, \lambda_i^j) x_m^k, y_i^j \rangle, \tag{3.39}$$

where

$$\Gamma(\lambda_m^k, \lambda_i^j) = I - e^{-\lambda_m^k} A_{-1} + \lambda_i^j \int_{-1}^0 e^{(\lambda_m^k - \lambda_i^j)\theta} \,\mathrm{d}\theta - \int_{-1}^0 e^{\lambda_m^k \theta} A_2(\theta) \,\mathrm{d}\theta$$

$$+ \int_{-1}^0 e^{(\lambda_m^k - \lambda_i^j)\theta} \int_0^\theta e^{\lambda_i^j s} A_3(s) \,\mathrm{d}s \,\mathrm{d}\theta + \lambda_i^j \int_{-1}^0 e^{(\lambda_m^k - \lambda_i^j)\theta} \int_0^\theta e^{\lambda_i^j s} A_2(s) \,\mathrm{d}s \,\mathrm{d}\theta. \tag{3.40}$$

The last two terms of $\Gamma(\lambda_m^k, \lambda_i^j)$ are integrals with the domain $-1 \le \theta \le s \le 0$ and we integrate first by s and after that by θ. Let us change the order of integrating: $\int_{-1}^0 (\int_0^\theta \,\mathrm{d}s) \,\mathrm{d}\theta = -\int_{-1}^0 (\int_\theta^0 \,\mathrm{d}s) \,\mathrm{d}\theta = -\int_{-1}^0 (\int_{-1}^s \,\mathrm{d}\theta) \,\mathrm{d}s$. Since also $\int_{-1}^0 e^{(\lambda_m^k - \lambda_i^j)\theta} \,\mathrm{d}\theta = \frac{1}{\lambda_m^k - \lambda_i^j}(1 - e^{\lambda_i^j - \lambda_m^k})$, we obtain

$$\int_{-1}^0 e^{(\lambda_m^k - \lambda_i^j)\theta} \int_0^\theta e^{\lambda_i^j s} A_i(s) \,\mathrm{d}s \,\mathrm{d}\theta = -\int_{-1}^0 e^{\lambda_i^j s} A_i(s) \int_{-1}^s e^{(\lambda_m^k - \lambda_i^j)\theta} \,\mathrm{d}\theta \,\mathrm{d}s$$

$$= \frac{1}{\lambda_m^k - \lambda_i^j} \left[e^{\lambda_i^j - \lambda_m^k} \int_{-1}^0 e^{\lambda_i^j s} A_i(s) \,\mathrm{d}s - \int_{-1}^0 e^{\lambda_m^k s} A_i(s) \,\mathrm{d}s \right].$$

Finally, we have

$$\Gamma(\lambda_m^k, \lambda_i^j) = \frac{1}{\lambda_m^k - \lambda_i^j} \Big[(\lambda_m^k - \lambda_i^j) I - (\lambda_m^k - \lambda_i^j) e^{-\lambda_m^k} A_{-1} + \lambda_i^j (1 - e^{\lambda_i^j - \lambda_m^k}) I$$

$$-(\lambda_m^k - \lambda_i^j) \int_{-1}^0 e^{\lambda_m^k \theta} A_2(\theta) \,\mathrm{d}\theta - \int_{-1}^0 e^{\lambda_m^k s} A_3(s) \,\mathrm{d}s - \lambda_i^j \int_{-1}^0 e^{\lambda_m^k s} A_2(s) \,\mathrm{d}s$$

$$+e^{\lambda_i^j-\lambda_m^k}\int_{-1}^{0} e^{\lambda_i^j s}A_3(s)\,\mathrm{d}s+\lambda_i^j e^{\lambda_i^j-\lambda_m^k}\int_{-1}^{0} e^{\lambda_i^j s}A_2(s)\,\mathrm{d}s\Bigg]$$

$$=\frac{1}{\lambda_m^k-\lambda_i^j}\Bigg[\lambda_m^k I-\lambda_m^k e^{-\lambda_m^k}A_{-1}-\int_{-1}^{0} e^{\lambda_m^k s}A_3(s)\,\mathrm{d}s-\lambda_m^k\int_{-1}^{0} e^{\lambda_m^k s}A_2(s)\,\mathrm{d}s$$

$$-\lambda_i^j e^{\lambda_i^j-\lambda_m^k}I+\lambda_i^j e^{-\lambda_m^k}A_{-1}+e^{\lambda_i^j-\lambda_m^k}\int_{-1}^{0} e^{\lambda_i^j s}A_3(s)\,\mathrm{d}s+\lambda_i^j e^{\lambda_i^j-\lambda_m^k}\int_{-1}^{0} e^{\lambda_i^j s}A_2(s)\,\mathrm{d}s\Bigg]$$

$$=\frac{1}{\lambda_m^k-\lambda_i^j}\left[-\triangle(\lambda_m^k)+e^{\lambda_i^j-\lambda_m^k}\triangle(\lambda_i^j)\right].$$

Taking into account that $x_m^k\in\mathrm{Ker}\triangle(\lambda_m^k)$, $y_i^j\in\mathrm{Ker}\triangle^*(\overline{\lambda_i^j})$ and $(\triangle(\lambda_i^j))^*=\triangle^*(\overline{\lambda_i^j})$, we conclude that

$$\langle\varphi_m^k,\psi_i^j\rangle_{M_2}=\left\langle\frac{1}{\lambda_m^k-\lambda_i^j}\left[-\triangle(\lambda_m^k)+e^{\lambda_i^j-\lambda_m^k}\triangle(\lambda_i^j)\right]x_m^k,y_i^j\right\rangle_{\mathbb{C}^n}=\frac{e^{\lambda_i^j-\lambda_m^k}}{\lambda_m^k-\lambda_i^j}\langle x_m^k,\triangle^*(\overline{\lambda_i^j})y_i^j\rangle_{\mathbb{C}^n}=0. \tag{3.41}$$

2. Let $(m,k)=(i,j)$, then from (3.39), (3.40) we have:

$$\langle\varphi_m^k,\psi_m^k\rangle=\langle\Gamma(\lambda_m^k)x_m^k,y_m^k\rangle,$$

where

$$\Gamma(\lambda_m^k)=I-e^{-\lambda_m^k}A_{-1}+\lambda_m^k I-\int_{-1}^{0} e^{\lambda_m^k\theta}A_2(\theta)\,\mathrm{d}\theta$$
$$+\int_{-1}^{0}\int_{0}^{\theta} e^{\lambda_m^k s}A_3(s)\,\mathrm{d}s\,\mathrm{d}\theta+\lambda_m^k\int_{-1}^{0}\int_{0}^{\theta} e^{\lambda_m^k s}A_2(s)\,\mathrm{d}s\,\mathrm{d}\theta. \tag{3.42}$$

The last two terms of $\Gamma(\lambda_m^k)$ are integrals with the domain $-1\le\theta\le s\le 0$ and we integrate first by s and after that by θ. Let us change the order of integrating: $\int_{-1}^{0}(\int_{0}^{\theta}\,\mathrm{d}s)\,\mathrm{d}\theta=-\int_{-1}^{0}(\int_{\theta}^{0}\,\mathrm{d}s)\,\mathrm{d}\theta=-\int_{-1}^{0}(\int_{-1}^{s}\,\mathrm{d}\theta)\,\mathrm{d}s$. Thus, we obtain

$$\Gamma(\lambda_m^k)=I-e^{-\lambda_m^k}A_{-1}+\lambda_m^k I-\int_{-1}^{0} e^{\lambda_m^k\theta}A_2(\theta)\,\mathrm{d}\theta$$

$$-\int_{-1}^{0} e^{\lambda_m^k s}A_3(s)\int_{-1}^{s}\,\mathrm{d}\theta\,\mathrm{d}s-\lambda_m^k\int_{-1}^{0} e^{\lambda_m^k s}A_2(s)\int_{-1}^{s}\,\mathrm{d}\theta\,\mathrm{d}s$$

$$=\left(I-e^{-\lambda_m^k}A_{-1}-\int_{-1}^{0} e^{\lambda_m^k\theta}A_2(\theta)\,\mathrm{d}\theta-\int_{-1}^{0} e^{\lambda_m^k s}sA_3(s)\,\mathrm{d}s-\lambda_m^k\int_{-1}^{0} e^{\lambda_m^k s}sA_2(s)\,\mathrm{d}s\right)$$

$$+\left(\lambda_m^k I-\int_{-1}^{0} e^{\lambda_m^k s}sA_3(s)\,\mathrm{d}s-\lambda_m^k\int_{-1}^{0} e^{\lambda_m^k s}sA_2(s)\,\mathrm{d}s\right)=\Gamma_1(\lambda_m^k)+\Gamma_2(\lambda_m^k).$$

It is easy to see that

$$\Gamma_1(\lambda_m^k)+\lambda_m^k e^{-\lambda_m^k}A_{-1}=-\triangle'(\lambda_m^k)$$

$$\Gamma_2(\lambda_m^k)-\lambda_m^k e^{-\lambda_m^k}A_{-1}=-\triangle(\lambda_m^k).$$

Taking into account that $x_m^k\in\mathrm{Ker}\triangle(\lambda_m^k)$, we conclude that

$$\langle\varphi_m^k,\psi_m^k\rangle_{M_2}=-\langle\triangle'(\lambda_m^k)x_m^k,y_m^k\rangle_{\mathbb{C}^n}. \tag{3.43}$$

The last completes the proof of the lemma. $\square$

Proof of Lemma 3.2. We choose $\|x_m^k\|_{\mathbb{C}^n} = 1$ and $\|y_m^k\|_{\mathbb{C}^n} = 1$ and obtain the following estimates.

$$\|\varphi_m^k\|^2 = \|(I - e^{-\lambda_m^k} A_{-1})x_m^k\|^2 + \int_{-1}^{0} \|e^{\lambda_m^k \theta} x_m^k\|^2 \,\mathrm{d}\theta \leq \|I - e^{-\lambda_m^k} A_{-1}\|^2 + \int_{-1}^{0} e^{2\mathrm{Re}\lambda_m^k \theta} \,\mathrm{d}\theta$$

$$\leq 1 + e^{2\mathrm{Re}\lambda_m^k}\|A_{-1}\|^2 + \frac{1 - e^{-2\mathrm{Re}\lambda_m^k}}{2\mathrm{Re}\lambda_m^k} \leq 1 + \|A_{-1}\|^2 + \frac{1 - e^{2r^{(N)}}}{2r^{(N)}} \leq C.$$

We have used here the fact that the real function $\frac{1-e^{-y}}{y}$ decreases monotone from ∞ to 1 when $y \to -0$.

Moreover,

$$\|\frac{1}{\overline{\lambda_m^k}}\psi_m^k\|^2 = \|y_m^k\|^2 + \int_{-1}^{0} \|(e^{-\overline{\lambda_m^k}\theta} - \frac{1}{\overline{\lambda_m^k}}A_2^*(\theta) + \frac{1}{\overline{\lambda_m^k}}e^{-\overline{\lambda_m^k}\theta}\int_0^\theta e^{\overline{\lambda_m^k}s}A_3^*(s)\,\mathrm{d}s$$

$$+e^{-\overline{\lambda_m^k}\theta}\int_0^\theta e^{\overline{\lambda_m^k}s}A_2^*(s)\,\mathrm{d}s)y_m^k\|^2\,\mathrm{d}\theta \leq \|y_m^k\|^2(1 + \frac{e^{2\mathrm{Re}\lambda_m^k} - 1}{2\mathrm{Re}\lambda_m^k} + \frac{1}{|\lambda_m^k|^2}\|A_2^*(\theta)\|^2_{L_2(-1,0;\mathbb{C}^{n\times n})}$$

$$+\frac{1}{|\lambda_m^k|^2}\int_{-1}^0 e^{-2\mathrm{Re}\lambda_m^k\theta}\,\mathrm{d}\theta\int_{-1}^0 e^{2\mathrm{Re}\lambda_m^k s}\|A_3^*(s)\|\,\mathrm{d}s + \int_{-1}^0 e^{-2\mathrm{Re}\lambda_m^k\theta}\,\mathrm{d}\theta\int_{-1}^0 e^{2\mathrm{Re}\lambda_m^k s}\|A_2^*(s)\|\,\mathrm{d}s)$$

$$\leq 1+\frac{1}{|\lambda_m^k|^2}\|A_2^*(\theta)\|^2_{L_2(-1,0;\mathbb{C}^{n\times n})}+\frac{e^{2\mathrm{Re}\lambda_m^k} - 1}{2\mathrm{Re}\lambda_m^k}\left(1 + \frac{1}{|\lambda_m^k|^2}\|A_3^*(\theta)\|^2_{L_2(-1,0;\mathbb{C}^{n\times n})} + \|A_2^*(\theta)\|^2_{L_2(-1,0;\mathbb{C}^{n\times n})}\right)$$

$$\leq 2 + \left(\frac{1}{|\lambda_m^k|^2} + 1\right)\|A_2^*(\theta)\|^2_{L_2(-1,0;\mathbb{C}^{n\times n})} + \frac{1}{|\lambda_m^k|^2}\|A_3^*(\theta)\|^2_{L_2(-1,0;\mathbb{C}^{n\times n})} \leq C.$$

Here we used the fact that the real function $\frac{e^y-1}{y}$ increases monotone from 0 to 1 when $y \to -0$. $\square$

Remark 3.2 *We note that the norm of eigenvectors ψ_m^k (with $\|y_m^k\| = 1$) increases infinitely when $k \to \infty$. This could be seen on example of eigenvectors $\widetilde{\psi}_m^k$ of the operator $\widetilde{\mathcal{A}}^*$ ($A_2(\theta) = A_2(\theta) \equiv 0$):*

$$\|\widetilde{\psi}_m^k\|^2 = \|y_m^k\|^2+\int_{-1}^0 \|\overline{\lambda_m^k}e^{-\overline{\lambda_m^k}\theta}y_m^k\|^2\,\mathrm{d}\theta = \|y_m^k\|^2\left(1 + |\lambda_m^k|^2\frac{e^{2\mathrm{Re}\lambda_m^k} - 1}{2\mathrm{Re}\lambda_m^k}\right) \geq (1+C|\lambda_m^k|^2) \to \infty.$$

Proof of Lemma 3.3. Let $\lambda_m^k \in \Lambda_1$ and we prove the representation (3.27).

From the form of the matrices R_m we conclude that $R_m^{-1} = R_m^* = R_m$, $1 \leq m \leq \ell_1$ and $R_1 = I$.

Let us analyze the matrix

$$\frac{1}{\lambda_m^k}R_m\triangle(\lambda_m^k)R_m = -I+e^{-\lambda_m^k}R_mA_{-1}R_m+\int_{-1}^0 e^{\lambda_m^k s}R_mA_2(s)R_m\,\mathrm{d}s+\frac{1}{\lambda_m^k}\int_{-1}^0 e^{\lambda_m^k s}R_mA_3(s)R_m\,\mathrm{d}s.$$

Since the matrix A_{-1} is in Jordan form (2.8), then, multiplying A_{-1} on R_m from the left and from the right we change places of one-dimensional Jordan blocks μ_1 and μ_m:

$$R_mA_{-1}R_m = \left(\begin{array}{cc} \mu_m & 0 \\ 0 & S \end{array}\right),$$

where $S \in C^{(n-1)\times(n-1)}$.

Let us consider the matrix $-I + e^{-\widetilde{\lambda}_m^{(k)}} R_m A_{-1} R_m$, where $\widetilde{\lambda}_m^k = \mathrm{i}(\arg \mu_m + 2\pi k)$ is an eigenvalue of the operator $\widetilde{\mathcal{A}}$, i.e. root of the equation $e^{\lambda} = \mu_m$. This matrix is of the form

$$-I + e^{-\widetilde{\lambda}_m^{(k)}} R_m A_{-1} R_m = \begin{pmatrix} 0 & 0 \\ 0 & -I_{n-1} + e^{-\widetilde{\lambda}_m^{(k)}} S \end{pmatrix},$$

where I_{n-1} is the identity matrix in $C^{(n-1)\times(n-1)}$. Thus, $\det(-I + e^{-\widetilde{\lambda}_m^{(k)}} R_m A_{-1} R_m) = 0$ and since the multiplicity of the eigenvalue $\mu_m \in \sigma_1$ equals 1, then $\mathrm{rg}(-I + e^{-\widetilde{\lambda}_m^{(k)}} R_m A_{-1} R_m) = n-1$, which means that $\det(-I_{n-1} + e^{-\widetilde{\lambda}_m^{(k)}} S) \neq 0$,

Since $|\lambda_m^k - \widetilde{\lambda}_m^{(k)}| \to 0$ when $k \to \infty$, then for any $\varepsilon > 0$ there exists $N > 0$ such that for $k: \ |k| \geq N$ the estimate $|e^{-\lambda_m^k} - e^{-\widetilde{\lambda}_m^{(k)}}| \|S\| \leq \varepsilon$ holds, and, thus, due to Proposition 4.2, we have

$$\det(-I_{n-1} + e^{-\lambda_m^k} S) = \det(-I_{n-1} + e^{-\widetilde{\lambda}_m^{(k)}} S + (e^{-\lambda_m^k} - e^{-\widetilde{\lambda}_m^{(k)}}) S) \neq 0. \tag{3.44}$$

According to Proposition 4.3, elements of the matrices $\int_{-1}^{0} e^{\lambda_m^k s} R_m A_i(s) R_m \, \mathrm{d}s$, $i = \overline{2,3}$ are small when $k \to \infty$. We introduce the following notation:

$$\int_{-1}^{0} e^{\lambda s} R_m A_2(s) R_m \, \mathrm{d}s + \frac{1}{\lambda} \int_{-1}^{0} e^{\lambda s} R_m A_3(s) R_m \, \mathrm{d}s = \begin{pmatrix} \varepsilon_{11}(\lambda) & \dots & \varepsilon_{1n}(\lambda) \\ \vdots & \ddots & \vdots \\ \varepsilon_{n1}(\lambda) & \dots & \varepsilon_{nn}(\lambda) \end{pmatrix}.$$

Let us choose $N > 0$ such that for any $k: \ |k| \geq N$ the estimate $|\varepsilon_{ij}(\lambda_m^k)| \leq \varepsilon$ holds. Therefore, due to Proposition 4.2, we obtain that

$$\det S_{m,k} \equiv \det \left(-I_{n-1} + e^{-\lambda_m^k} S + \begin{pmatrix} \varepsilon_{22}(\lambda_m^k) & \dots & \varepsilon_{2n}(\lambda_m^k) \\ \vdots & \ddots & \vdots \\ \varepsilon_{n2}(\lambda_m^k) & \dots & \varepsilon_{nn}(\lambda_m^k) \end{pmatrix} \right) \neq 0. \tag{3.45}$$

In the notation introduced the singular matrix $\frac{1}{\lambda_m^k} R_m \triangle(\lambda_m^k) R_m$ has the following structure:

$$\frac{1}{\lambda_m^k} R_m \triangle(\lambda_m^k) R_m = \begin{pmatrix} -1 + e^{-\lambda_m^k}\mu_m + \varepsilon_{11}(\lambda_m^k) & \varepsilon_{12}(\lambda_m^k) \ \dots \ \varepsilon_{1n}(\lambda_m^k) \\ \varepsilon_{21}(\lambda_m^k) & \\ \vdots & S_{m,k} \\ \varepsilon_{n1}(\lambda_m^k) & \end{pmatrix}. \tag{3.46}$$

We denote elements of $S_{m,k}$ as $S_{m,k} = \{s_{ij}^{m,k}\}_{2 \leq i,j \leq n} = \{s_{ij}\}_{2 \leq i,j \leq n}$. Since $\det S_{m,k} \neq 0$ then the first row of the matrix (3.46) is a linear combination of all other rows and also the first column is a linear combination of all other columns:

$$\varepsilon_{1i}(\lambda_m^k) = p_2 s_{2i} + \ldots + p_n s_{ni}, \quad i = \overline{2,n} \tag{3.47}$$

and

$$\varepsilon_{j1}(\lambda_m^k) = q_2 s_{j2} + \ldots + q_n s_{jn}, \quad j = \overline{2,n}. \tag{3.48}$$

The last relations defines two matrices given by (3.26), i.e.

$$P_{m,k} = \begin{pmatrix} 1 & -p_2 & \dots & -p_n \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \quad Q_{m,k} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ -q_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -q_n & 0 & \dots & 1 \end{pmatrix},$$

and direct computations gives us that the matrix $\frac{1}{\lambda_m^k} P_{m,k} R_m \triangle(\lambda_m^k) R_m Q_{m,k}$ has the form (3.27), i.e.:

$$\frac{1}{\lambda_m^k} P_{m,k} R_m \triangle(\lambda_m^k) R_m Q_{m,k} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & S_{m,k} & \\ 0 & & & \end{pmatrix}.$$

Now let us give the estimate (3.28), i.e. an estimate on numbers p_i and q_i. The equations (3.47), (3.48) can be rewritten in the form:

$$v_1 = (S_{m,k})^T w_1, \qquad v_2 = S_{m,k} w_2,$$

where $v_1 = (\varepsilon_{12}(\lambda_m^k), \dots, \varepsilon_{1n}(\lambda_m^k))^T$, $w_1 = (p_2, \dots, p_n)^T$, $v_2 = (\varepsilon_{21}(\lambda_m^k), \dots, \varepsilon_{n1}(\lambda_m^k))^T$, $w_2 = (q_2, \dots, q_n)^T$. Since $\det S_{m,k} \neq 0$ we have:

$$w_1 = (S_{m,k}^{-1})^T v_1, \qquad w_2 = S_{m,k}^{-1} v_2$$

and let us estimate $\|S_{m,k}^{-1}\|$.

From (3.44) and (3.45) we see that $S_{m,k} = -I_{n-1} + \frac{1}{\mu_m} S - B_{m,k}$, where elements of the matrix $B_{m,k}$ are small: $|b_{ij}| < \varepsilon$. Thus, there exists $N \in \mathbb{Z}$ such that the norm of the operator $\widetilde{B}_{m,k} \equiv (-I_{n-1} + \frac{1}{\mu_m} S)^{-1} B_{m,k}$ is small: $\|\widetilde{B}_{m,k}\| < 1$ (and, moreover, it is less then any $\varepsilon > 0$) for any $|k| > N$. Thus, there exists an inverse of the matrix $I_{n-1} - \widetilde{B}_{m,k}$ (see e.g. [13, p.233, Theorem 5]), which is uniformly bounded for any k (for example, we take $\|\widetilde{B}_{m,k}\| < \frac{1}{2}$):

$$\|(I_{n-1} - \widetilde{B}_{m,k})^{-1}\| = \|\sum_{i=0}^{\infty} (\widetilde{B}_{m,k})^i\| \le C_1.$$

Thus, we have

$$\|S_{m,k}^{-1}\| = \|(I_{n-1} - \widetilde{B}_{m,k})^{-1}(-I_{n-1} + \frac{1}{\mu_m} S)^{-1}\| \le C_1 \|(-I_{n-1} + \frac{1}{\mu_m} S)^{-1}\| \le C.$$

Since $\varepsilon_{1i}(\lambda_m^k)$ and $\varepsilon_{j1}(\lambda_m^k)$ are small then $\|v_1\| \le \varepsilon$ and $\|v_2\| \le \varepsilon$, therefore, $\|w_1\| \le \varepsilon C$, $\|w_2\| \le \varepsilon C$ and we conclude that the estimate (3.28) holds. $\square$

Proof of Corollary 3.1. The matrix function $\widehat{\triangle}_{m,k}(\lambda) \stackrel{\text{def}}{=} \frac{1}{\lambda} P_{m,k} R_m \triangle(\lambda) R_m Q_{m,k}$ is analytic in a neighborhood of the point λ_m^k since $\triangle(\lambda)$ is analytic.

Moreover, since also $\widehat{\triangle}_{m,k}(\lambda_m^k)$ is of the form (3.27), then we conclude that $\triangle(\lambda)$ is of the form (3.29), i.e.

$$\widehat{\triangle}_{m,k}(\lambda) = \begin{pmatrix} (\lambda - \lambda_m^k) r_{11}(\lambda) & (\lambda - \lambda_m^k) r_{12}(\lambda) & \dots & (\lambda - \lambda_m^k) r_{1n}(\lambda) \\ (\lambda - \lambda_m^k) r_{21}(\lambda) & & & \\ \vdots & & M_{m,k}(\lambda) & \\ (\lambda - \lambda_m^k) r_{n1}(\lambda) & & & \end{pmatrix}, \qquad \lambda \in U(\lambda_m^k),$$

where $r_{ij}(\lambda) = r_{ij}^{m,k}(\lambda)$ are analytic functions.

Let us prove the relation (3.30). If we suppose that $r_{11}^{m,k}(\lambda_m^k) = 0$ then $(\lambda - \lambda_m^k) r_{1i}^{m,k}(\lambda) = (\lambda - \lambda_m^k)^2 \widetilde{r}_{11}^{m,k}(\lambda)$. Decomposing $\det \widehat{\triangle}_{m,k}(\lambda)$ by the elements of the first row, we see that all the term of this decomposition have the multiplier $(\lambda - \lambda_m^k)^2$, i.e. $\det \widehat{\triangle}_{m,k}(\lambda) = (\lambda - \lambda_m^k)^2 r(\lambda)$, where $r(\lambda)$ is an analytic function. The last contradicts to the assumption that λ_m^k is an eigenvalue of the multiplicity one of the operator $\mathcal{A}$.

Moreover, taking into account (3.46) and the form of the transformations (3.27) we see that

$$(\lambda-\lambda_m^k)r_{11}^k(\lambda)=(-1+e^{-\lambda}\mu_m+\varepsilon_{11}(\lambda))-\sum_{i=2}^{n}p_i\varepsilon_{i1}(\lambda)-\sum_{j=2}^{n}q_j\varepsilon_{1j}(\lambda)$$

and, therefore,

$$r_{11}(\lambda_m^k)=((\lambda-\lambda_m^k)r_{11}(\lambda))'_{\lambda=\lambda_m^k}=-e^{-\lambda_m^k}\mu_m+\left(\varepsilon_{11}(\lambda)-\sum_{i=2}^{n}p_i\varepsilon_{i1}(\lambda)-\sum_{j=2}^{n}q_j\varepsilon_{1j}(\lambda)\right)'_{\lambda=\lambda_m^k}.$$

The terms $(\varepsilon_{ij}(\lambda))'$ are of the form

$$\varepsilon_{ij}(\lambda))'=\int_{-1}^{0}e^{\lambda s}s(A_2(s))_{ij}\,\mathrm{d}s+\frac{1}{\lambda}\int_{-1}^{0}e^{\lambda s}s(A_3(s))_{ij}\,\mathrm{d}s-\frac{1}{\lambda^2}\int_{-1}^{0}e^{\lambda s}(A_3(s))_{ij}\,\mathrm{d}s,$$

therefore, using Proposition 4.3, we conclude that

$$\left(\varepsilon_{11}(\lambda)-\sum_{i=2}^{n}p_i\varepsilon_{i1}(\lambda)-\sum_{j=2}^{n}q_j\varepsilon_{1j}(\lambda)\right)'_{\lambda=\lambda_m^k}\to 0,\quad k\to\infty.$$

Since $-e^{-\lambda_m^k}\mu_m\to -1$ when $k\to\infty$, then we obtain the relation (3.30) and, in particular, there exists a constant $C>0$ and an integer N such that for $|k|>N$ we have

$$0<C\leq |r_{11}^k(\lambda_k)|.$$

The last completes the proof of the proposition. □

Proof of Lemma 3.4. Let us first prove the estimate (3.32) for eigenvalues $\lambda_1^k\in\Lambda_1$. We use Lemma 3.3: since $x_1^k\in \mathrm{Ker}\triangle(\lambda_1^k)$, then

$$0=\frac{1}{\lambda_1^k}P_{1,k}^{-1}P_{1,k}\triangle(\lambda_1^k)Q_{1,k}Q_{1,k}^{-1}x_1^k=P_{1,k}^{-1}\widehat{\triangle}(\lambda_1^k)Q_{1,k}^{-1}x_1^k,\tag{3.49}$$

where $P_{1,k}$, $Q_{1,k}$, $\widehat{\triangle}(\lambda_1^k)$ are defined by (3.26), (3.27). Taking into account the form (3.27) of the matrix $\widehat{\triangle}(\lambda_1^k)$ and, namely, $\det S_{1,k}\neq 0$, we conclude that $Q_{1,k}^{-1}x_1^k=(\widehat{x}_1,0,\ldots,0)^T$. On the other hand, from (3.31) and multiplying directly $Q_{1,k}^{-1}$ on x_1^k, we conclude that $\widehat{x}_1=(x_1^k)_1$ and $(x_1^k)_i=-q_i(x_1^k)_1$, $i=\overline{2,n}$. Thus, taking into account (3.28), we have:

$$1=\|x_1^k\|^2=|(x_1^k)_1|^2(1+|q_2|^2+\ldots+|q_n|^2)\leq |(x_1^k)_1|^2(1+(n-1)\varepsilon^2)$$

and, therefore, we conclude that $|(x_1^k)_1|\geq\frac{1}{\sqrt{1+(n-1)\varepsilon^2}}\to 1$ when $k\to\infty$. Summarizing, we have

$$Q_{1,k}^{-1}x_1^k=((x_1^k)_1,0,\ldots,0)^T,\qquad 0<C\leq|(x_1^k)_1|\leq 1,\lambda_1^k\in\Lambda_1.\tag{3.50}$$

Conjugating (3.27), we obtain

$$\begin{pmatrix} 0 & 0 \ \ldots & 0\\ 0 & & \\ \vdots & S_{1,k}^* & \\ 0 & & \end{pmatrix}=(\frac{1}{\lambda_1^k}P_{1,k}\triangle(\lambda_1^k)Q_{1,k})^*=\frac{1}{\overline{\lambda_1^k}}Q_{1,k}^*\triangle^*(\overline{\lambda_1^k})P_{1,k}^*=\widehat{\triangle}^*(\overline{\lambda_1^k}).\tag{3.51}$$

Since $y_1^k \in \mathrm{Ker}\triangle^*(\overline{\lambda_1^k})$, then

$$0 = \frac{1}{\overline{\lambda_1^k}}(Q_{1,k}^*)^{-1}Q_{1,k}^*\triangle(\lambda_1^k)P_{1,k}^*(P_{1,k}^*)^{-1}y_1^k = (Q_{1,k}^*)^{-1}\widehat{\triangle}^*(\overline{\lambda_1^k})(P_{1,k}^*)^{-1}y_1^k \tag{3.52}$$

and, taking into account the form (3.51) of the matrix $\widehat{\triangle}^*(\overline{\lambda_1^k})$, we conclude that $(P_{1,k}^*)^{-1}y_1^k = (\widehat{y}_1, 0, \ldots, 0)^T$. On the other hand, from the form of $(P_{1,k}^*)^{-1}$ we conclude that $\widehat{y}_1 = (y_1^k)_1$ and $(y_1^k)_i = -\overline{p}_i(y_1^k)_1$, $i = \overline{2, n}$. Thus, taking into account (3.28), we have:

$$1 = \|y_1^k\|^2 = |(y_1^k)_1|^2(1 + |\overline{p}_2|^2 + \ldots + |\overline{p}_n|^2) \le |(y_1^k)_1|^2(1 + (n-1)\varepsilon^2)$$

and, therefore, we conclude that $|(y_1^k)_1| \ge \frac{1}{\sqrt{1+(n-1)\varepsilon^2}} \to 1$ when $k \to \infty$.

$$(P_{1,k}^*)^{-1}y_1^k = ((y_1^k)_1, 0, \ldots, 0)^T, \qquad 0 < C \le |(y_1^k)_1| \le 1, \lambda_1^k \in \Lambda_1. \tag{3.53}$$

Differentiating (3.29) and putting $\lambda = \lambda_1^k$ we obtain

$$\begin{pmatrix} r_{11}(\lambda_1^k) & r_{12}(\lambda_1^k) & \ldots & r_{1n}(\lambda_1^k) \\ r_{21}(\lambda_1^k) & & & \\ \vdots & & M'_{m,k}(\lambda_1^k) & \\ r_{n1}(\lambda_1^k) & & & \end{pmatrix} = \widehat{\triangle}'(\lambda_1^k) = \left(\frac{1}{\overline{\lambda_1^k}}P_{1,k}\triangle(\overline{\lambda_1^k})Q_{1,k}\right)'_{\lambda=\lambda_1^k}$$

$$= P_{1,k}\left(\frac{1}{\lambda_1^k}\triangle'(\lambda_1^k) - \frac{1}{(\lambda_1^k)^2}\triangle(\lambda_1^k)\right)Q_{1,k}. \tag{3.54}$$

Finally, we have

$$-\frac{1}{\lambda_1^k}\langle\triangle'(\lambda_1^k)x_1^k, y_1^k\rangle = -\langle P_{1,k}^{-1}P_{1,k}\left(\frac{1}{\lambda_1^k}\triangle'(\lambda_1^k) - \frac{1}{(\lambda_1^k)^2}\triangle(\lambda_1^k)\right)Q_{1,k}Q_{1,k}^{-1}x_1^k, y_1^k\rangle$$

$$= -\langle P_{1,k}\left(\frac{1}{\lambda_1^k}\triangle'(\lambda_1^k) - \frac{1}{(\lambda_1^k)^2}\triangle(\lambda_1^k)\right)Q_{1,k}Q_{1,k}^{-1}x_1^k, (P_{1,k}^{-1})^*y_1^k\rangle$$

$$= -\langle \begin{pmatrix} r_{11}(\lambda_1^k) & r_{12}(\lambda_1^k) & \ldots & r_{1n}(\lambda_1^k) \\ r_{21}(\lambda_1^k) & & & \\ \vdots & & M'_{m,k}(\lambda_1^k) & \\ r_{n1}(\lambda_1^k) & & & \end{pmatrix}\begin{pmatrix} (x_1^k)_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} (y_1^k)_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\rangle$$

$$= -r_{11}(\lambda_1^k)(x_1^k)_1(y_1^k)_1 \to 1, \qquad k \to \infty. \tag{3.55}$$

From the last the estimate (3.32) follows and the lemma is proved in the case of eigenvalue $\lambda_1^k \in L_1^{(m)}$. □

Remark 3.3 *For $\lambda_m^k \in \Lambda_1$, $1 \le m \le \ell_1$ we have that*

$$-\frac{1}{\lambda_m^k}\langle\triangle'(\lambda_m^k)x_m^k, y_m^k\rangle = -r_{11}(\lambda_m^k)(x_m^k)_m(y_m^k)_m \to 1, \qquad k \to \infty. \tag{3.56}$$

Proof of Remark 3.3. The ideas of our arguing remain quite the same as in the case when we consider $\lambda_1^k \in \Lambda_1$ but now arguing becomes more complicated. As we have noted in the proof of Lemma 3.3, multiplying A_{-1} on R_m from the left and from the right we change places of one-dimensional Jordan blocks μ_1 and μ_j. Therefore, all previous arguing could be applying to the matrix $R_m\triangle(\lambda_m^k)R_m$ instead of $\triangle(\lambda_m^k)$. Also, we note that $R_m^{-1}=R_m^*=R_m$, $1\le m\le \ell_1$ and $R_1=I$.

Using Lemma 3.3 and since $x_m^k\in \operatorname{Ker}\triangle(\lambda_m^k)$, we have

$$0=\frac{1}{\lambda_m^k}R_mP_{m,k}^{-1}P_{m,k}R_m\triangle(\lambda_m^k)R_mQ_{m,k}Q_{m,k}^{-1}R_mx_m^k=R_mP_{m,k}^{-1}\widehat{\triangle}(\lambda_m^k)Q_{m,k}^{-1}R_mx_m^k, \tag{3.57}$$

where $P_{m,k}$, $Q_{m,k}$, $\widehat{\triangle}(\lambda_m^k)$ are of the form (3.26), (3.27). From the form (3.27) of the matrix $\widehat{\triangle}(\lambda_m^k)$ we conclude that $Q_{m,k}^{-1}R_mx_m^k=(\widehat{x}_1,0,\ldots,0)^T$. Multiplication $Q_{m,k}^{-1}$ on R_m from the right changes places the first and the j-th column of $Q_{m,k}^{-1}$, therefore, multiplying $Q_{m,k}^{-1}R_m$ on x_m^k, we obtain: $\widehat{x}_1=(x_m^k)_m$ and $(x_m^k)_i=-q_m(x_m^k)_m$, $i\in\{1,\ldots,n\}\backslash\{m\}$. Thus, taking into account (3.28), we have:

$$1=\|x_m^k\|^2\le |(x_m^k)_m|^2(1+(n-1)\varepsilon^2)$$

and, therefore, we conclude that $|(x_m^k)_m|\ge \frac{1}{\sqrt{1+(n-1)\varepsilon^2}}\to 1$ when $k\to\infty$. Summarizing, we have

$$Q_{m,k}^{-1}R_mx_m^k=((x_m^k)_m,0,\ldots,0)^T,\qquad 0<C\le|(x_m^k)_m|\le 1, \lambda_m^k\in\Lambda_1. \tag{3.58}$$

The same arguing gives us that

$$(P_{m,k}^*)^{-1}R_my_m^k=((y_m^k)_m,0,\ldots,0)^T,\qquad 0<C\le|(y_m^k)_m|\le 1, \lambda_m^k\in\Lambda_1. \tag{3.59}$$

and also

$$\begin{pmatrix} r_{11}(\lambda_m^k) & r_{12}(\lambda_m^k) & \ldots & r_{1n}(\lambda_m^k)\\ r_{21}(\lambda_m^k) & & & \\ \vdots & & M'_{m,k}(\lambda_m^k) & \\ r_{n1}(\lambda_m^k) & & & \end{pmatrix}=P_{m,k}R_m\left(\frac{1}{\lambda_m^k}\triangle'(\lambda_m^k)-\frac{1}{(\lambda_m^k)^2}\triangle(\lambda_m^k)\right)R_mQ_{m,k}. \tag{3.60}$$

Finally, we obtain

$$\begin{aligned}
-\frac{1}{\lambda_m^k}\langle\triangle'(\lambda_m^k)x_m^k,y_m^k\rangle &= -\langle R_mP_{m,k}^{-1}P_{m,k}R_m\left(\frac{1}{\lambda_m^k}\triangle'(\lambda_m^k)-\frac{1}{(\lambda_m^k)^2}\triangle(\lambda_m^k)\right)R_mQ_{m,k}Q_{m,k}^{-1}R_mx_m^k,y_m^k\rangle\\
&= -\langle P_{m,k}R_m\left(\frac{1}{\lambda_m^k}\triangle'(\lambda_m^k)-\frac{1}{(\lambda_m^k)^2}\triangle(\lambda_m^k)\right)R_mQ_{m,k}Q_{m,k}^{-1}x_m^k,(P_{m,k}^{-1})^*R_my_m^k\rangle\\
&= -\left\langle\begin{pmatrix} r_{11}(\lambda_m^k) & r_{12}(\lambda_m^k) & \ldots & r_{1n}(\lambda_m^k)\\ r_{21}(\lambda_m^k) & & & \\ \vdots & & M'_{m,k}(\lambda_m^k) & \\ r_{n1}(\lambda_m^k) & & & \end{pmatrix}\begin{pmatrix}(x_m^k)_m\\0\\\vdots\\0\end{pmatrix},\begin{pmatrix}(y_m^k)_m\\0\\\vdots\\0\end{pmatrix}\right\rangle\\
&= -r_{11}(\lambda_m^k)(x_m^k)_m(y_m^k)_m\to 1,\qquad k\to\infty.
\end{aligned} \tag{3.61}$$

The last completes the proof of the lemma. $\square$

4 Resolvent boundedness

Our aim is to prove that the resolvent $R(\lambda, \mathcal{A})$ is uniformly bounded on the subspace M_2^0 for $\lambda :\ \mathrm{Re}\lambda \geq 0$. We give the proof of this proposition by the following steps:

1. We prove that the relation $D(z, \xi, \lambda_m^k) \in \mathrm{Im}\triangle(\lambda_m^k)$ holds for any $(z, \xi(\theta))^T \in M_2^0$ and for any $\lambda_m^k \in \Lambda_1$.

2. Using the previous fact we prove that the resolvent is uniformly bounded on the subspace M_2^0 in the neighborhood of each $\lambda_m^k \in \Lambda_1$.

3. Finally, we prove that the resolvent is uniformly bounded on the subspace M_2^0 in the whole right half-plane.

We need the following propositions.

Lemma 4.1 *For any vector $g = (z, \xi(\theta))^T \in M_2^0$ and for any eigenvalue $\lambda_m^k \in \Lambda_1$ the following relation holds:*

$$D(z, \xi, \lambda_m^k) \in \mathrm{Im}\triangle(\lambda_m^k). \tag{4.62}$$

Lemma 4.2 *The vector-function $\triangle^{-1}(\lambda)D(z, \xi, \lambda)$ is uniformly bounded in neighborhoods $U_\delta(\lambda_m^k)$ of eigenvalues $\lambda_m^k \in \Lambda_1$ for some fixed $\delta > 0$, i.e.:*

1. for any $1 \leq m \leq \ell_1$, $|k| > N$ the estimate $\|\triangle^{-1}(\lambda)D(z, \xi, \lambda)\| \leq C_{m,k}$ holds in a neighborhood $U_\delta(\lambda_m^k)$;

2. there exists $C > 0$ such that $C_{m,k} \leq C$ for all $1 \leq m \leq \ell_1$, $|k| > N$.

Lemma 4.3 *For each $(z, \xi) \in M_2$ there exist constants C_1, C_2 such that for any $\lambda :\ \mathrm{Re}\lambda > 0$ we have*

$$\left\| \frac{1}{\lambda}\triangle(\lambda) \right\| \leq C_1, \tag{4.63}$$

$$\left\| \frac{1}{\lambda}D(z, \xi, \lambda) \right\| \leq C_2. \tag{4.64}$$

Proposition 4.1 *If the vector $y \in \mathrm{Im}A$, $A \in \mathbb{C}^{n\times n}$, then for any two matrices P, Q such that $\det P \neq 0$ and $\det Q \neq 0$ the relation $Py \in \mathrm{Im}(PAQ)$ holds.*

Proposition 4.2 *For any matrix $A \in \mathbb{C}^{n\times n}$ with $\det A \neq 0$ there exists $\varepsilon_0 > 0$ such that for any matrix $B = \{b_{ij}\} \in \mathbb{C}^{n\times n}$: $|b_{ij}| < \varepsilon_0$ we have*

$$\det(A + B) \neq 0.$$

Proposition 4.3 *Let $L_0 \subset \mathbb{C}$ be a bounded closed set and $f(s) \in L_2[-1, 0]$. Denote by $a_k(\lambda) = \int_{-1}^{0} e^{2\pi iks} e^{\lambda s} f(s)\, \mathrm{d}s$, $\lambda \in L_0$, $k \in \mathbb{Z}$. Then $|a_k(\lambda)| \to 0$ when $k \to \infty$ uniformly on the set L_0.*

Corollary 4.1 *If the sequence $\{\lambda_k\}$ is such that $\mathrm{Im}\lambda_k \to \infty$ and $-\infty < a \leq \mathrm{Re}\lambda_k \leq b < \infty$ then for any $f(s) \in L_2(0, 1; \mathbb{C}^{n\times m})$ we have: $\int_{-1}^{0} e^{\lambda_k s} f(s)\, \mathrm{d}s \to 0$ when $k \to \infty$.*

Proof of Theorem 2.2. We introduce two sets $K_1(\delta) = \{\lambda : \mathrm{Re}\lambda \geq 0\} \cap U_\delta(\widetilde{\lambda}_m^k)$ and $K_2(\delta) = \{\lambda : \mathrm{Re}\lambda \geq 0\} \backslash K_1$ for small enough $\delta > 0$ such that Lemma 4.2 holds.

First, let us prove that $\|R(\lambda, \mathcal{A})x\| \leq C\|x\|$ for any $\lambda \in K_1(\delta)$ and $x \in M_2^0$. Due to Lemma 4.2 we have: $\|\triangle^{-1}(\lambda)D(z, \xi, \lambda)\| \leq C_1$. Due to Corollary 4.1 we obtain that $\| \int_{-1}^0 e^{-\lambda s}\xi(s)\, \mathrm{d}s\| \leq C_2$. Thus, for any $x = (z, \xi(\cdot)) \in M_2^0$ we obtain:

$$\|R(\lambda, \mathcal{A})x\| = \|e^{-\lambda} A_{-1} \int_{-1}^0 e^{-\lambda s}\xi(s)\, \mathrm{d}s + (I - e^{-\lambda}A_{-1})\triangle^{-1}(\lambda)D(z, \xi, \lambda)\|_{\mathbb{C}^n}$$

$$+\| \int_0^\theta e^{\lambda(\theta - s)}\xi(s)\, \mathrm{d}s + e^{\lambda\theta}\triangle^{-1}(\lambda)D(z, \xi, \lambda)\|_{L_2}$$

$$\leq e^\delta\|A_{-1}\|C_2 + (1 + e^\delta\|A_{-1}\|)C_1 + (\int_{-1}^0 \| \int_0^\theta e^{\lambda(\theta-s)}\xi(s)\, \mathrm{d}s + e^{\lambda\theta}\triangle^{-1}(\lambda)D(z, \xi, \lambda)\|_{\mathbb{C}^n}^2\, \mathrm{d}\theta)^{\frac{1}{2}}$$

$$\leq e^\delta\|A_{-1}\|C_2 + (1 + e^\delta\|A_{-1}\|)C_1 + (e^\delta C_2 + C_1^2)^{\frac{1}{2}} \leq C.$$

Thus, for any $x =\in M_2^0$ the family $R(\lambda, \mathcal{A})x$ is bounded and due to Banach-Steinhaus theorem we have that $R(\lambda, \mathcal{A})$ uniformly bounded on M_2^0.

Now we prove that $\|R(\lambda, \mathcal{A})x\| \leq C\|x\|$ for any $\lambda \in K_2(\delta)$ and $x \in M_2$.

We begin with proof that $|\frac{1}{\lambda} \det \triangle(\lambda)| \geq \varepsilon$ for some $\varepsilon > 0$ and for any $\lambda \in K_2(\delta)\backslash U(0)$. Let us suppose the contrary then there exists a sequence $\{\lambda_i\}$ such that $|\frac{1}{\lambda_i} \det \triangle(\lambda_i)| \to 0$. If we suppose that this sequence is bounded then its subsequence converges: $\lambda_{i_j} \to \widehat{\lambda}$ and $|\frac{1}{\widehat{\lambda}} \det \triangle(\widehat{\lambda})| = 0$. However, closure of $\lambda \in K_2(\delta)\backslash U(0)$ does not contains zeroes of $\det \triangle(\lambda)$ and we obtain a contradiction. Thus, $|\lambda_i| \to \infty$ when $i \to \infty$. Analyzing the matrix $\frac{1}{\lambda}\triangle(\lambda) = -I + e^{-\lambda}A_{-1} + \int_{-1}^0 e^{\lambda s}A_2(s)\, \mathrm{d}s + \frac{1}{\lambda}\int_{-1}^0 e^{\lambda s}A_3(s)\, \mathrm{d}s$ we see that $\int_{-1}^0 e^{\lambda s}A_2(s)\, \mathrm{d}s \to 0$ and $\frac{1}{\lambda}\int_{-1}^0 e^{\lambda s}A_3(s)\, \mathrm{d}s \to 0$ and $|\det(-I + e^{-\lambda}A_{-1})| = |\prod(1 + e^{-\lambda}\mu_m)| \geq \prod(1 + e^\delta\mu_m)$. Thus, we obtain a contradiction again.

Since also $\|\frac{1}{\lambda}\triangle(\lambda)\| \leq C_1$ and $\left\|\frac{1}{\lambda}D(z, \xi, \lambda)\right\| \leq C_2$ we obtain that $\|\triangle^{-1}(\lambda)D(z, \xi, \lambda)\| \leq C_3$, for all $\lambda \in K_2(\delta)\backslash U(0)$. Since $|e^{-\lambda}\int_{-1}^0 s^k e^{-\lambda s}\, \mathrm{d}s| \leq C_4$ for all $\lambda \in K_2(\delta)\backslash U(0)$ and $\{s^k\}$ is a dense set in L_2 we obtain that $\|e^{-\lambda}\int_{-1}^0 e^{-\lambda s}\xi(s)\, \mathrm{d}s\| \leq C_5$. Similarly we obtain an estimate

$$\|R(\lambda, \mathcal{A})x\| \leq C$$

for any $\lambda \in K_2(\delta)$ and $x \in M_2$ and thus, due to Banach-Steinhaus theorem we have that $R(\lambda, \mathcal{A})$, $\lambda \in K_2(\delta)\backslash U(0)$ uniformly bounded on M_2.

The last completes the proof of the theorem.

Proof of Lemma 4.1. We show that $D(z, \xi, \lambda_m^k) \perp \mathrm{Ker}\triangle^*(\overline{\lambda_m^k})$ for any vector $g = \begin{pmatrix} z \\ \xi(\theta) \end{pmatrix} \in M_2^0$ and any $\lambda_m^k \in \Lambda_1$.

Let $\psi_m^k \in \widehat{M}_2^1$ be an eigenvector of $\mathcal{A}^*$ corresponding to the eigenvalue $\overline{\lambda_m^k}$. Thus, this eigenvector is of the form (3.23), i.e.

$$\psi_m^k = \begin{pmatrix} y_m^k \\ \left[\overline{\lambda_m^k} e^{-\overline{\lambda_m^k}\theta} - A_2^*(\theta) + e^{-\overline{\lambda_m^k}\theta}\int_0^\theta e^{\overline{\lambda_m^k}s}A_3^*(s)\, \mathrm{d}s + \overline{\lambda_m^k}e^{-\overline{\lambda_m^k}\theta}\int_0^\theta e^{\overline{\lambda_m^k}s}A_2^*(s)\, \mathrm{d}s\right] y_m^k \end{pmatrix},$$

where $y_m^k \in \mathrm{Ker}\triangle^*(\overline{\lambda_m^k})$. We use the orthogonality of ψ_m^k and g:

$$0 = \langle g, \psi_m^k\rangle = \langle z, y_m^k\rangle + \int_{-1}^0 \langle \xi(\theta), \overline{\lambda_m^k}e^{-\overline{\lambda_m^k}\theta}y_m^k\rangle\, \mathrm{d}\theta - \int_{-1}^0 \langle \xi(\theta), A_2^*(\theta)y_m^k\rangle\, \mathrm{d}\theta$$

$$+\int_{-1}^{0}\langle\xi(\theta), e^{-\overline{\lambda_m^k}\theta}\int_0^\theta e^{\overline{\lambda_m^k}s}A_3^*(s)\ \mathrm{d}s\cdot y_m^k\rangle\ \mathrm{d}\theta+\int_{-1}^{0}\langle\xi(\theta),\overline{\lambda_m^k}e^{-\overline{\lambda_m^k}\theta}\int_0^\theta e^{\overline{\lambda_m^k}s}A_2^*(s)\ \mathrm{d}s\cdot y_m^k\rangle\ \mathrm{d}\theta$$

$$=\langle z,y_m^k\rangle+\langle\int_{-1}^{0}\lambda_m^k e^{-\lambda_m^k\theta}\xi(\theta)\ \mathrm{d}\theta, y_m^k\rangle-\langle\int_{-1}^{0}A_2(\theta)\xi(\theta)\ \mathrm{d}\theta, y_m^k\rangle$$

$$+\langle\int_{-1}^{0}\left[e^{-\lambda_m^k\theta}\int_0^\theta e^{\lambda_m^k s}A_3(s)\ \mathrm{d}s\right]\xi(\theta)\ \mathrm{d}\theta, y_m^k\rangle+\langle\int_{-1}^{0}\left[\lambda_m^k e^{-\lambda_m^k\theta}\int_0^\theta e^{\lambda_m^k s}A_2(s)\ \mathrm{d}s\right]\xi(\theta)\ \mathrm{d}\theta, y_m^k\rangle.$$

The last two terms are integrals with the domain $-1\le\theta\le s\le 0$ and we integrate first by s and after that by θ. Let us change the order of integrating: $\int_{-1}^{0}(\int_0^\theta\ \mathrm{d}s)\ \mathrm{d}\theta = -\int_{-1}^{0}(\int_\theta^0\ \mathrm{d}s)\ \mathrm{d}\theta=-\int_{-1}^{0}(\int_{-1}^{s}\ \mathrm{d}\theta)\ \mathrm{d}s$. Thus, we obtain

$$\langle g,\psi_m^k\rangle=\langle z+\int_{-1}^{0}\lambda_m^k e^{-\lambda_m^k\theta}\xi(\theta)\ \mathrm{d}\theta+\int_{-1}^{0}A_2(\theta)\xi(\theta)\ \mathrm{d}\theta$$

$$-\int_{-1}^{0}e^{\lambda_m^k s}A_3(s)\int_{-1}^{s}e^{-\lambda_m^k\theta}\xi(\theta)\ \mathrm{d}\theta\ \mathrm{d}s-\int_{-1}^{0}\lambda e^{\lambda_m^k s}A_2(s)\int_{-1}^{s}e^{-\lambda_m^k\theta}\xi(\theta)\ \mathrm{d}\theta\ \mathrm{d}s, y_m^k\rangle.\qquad(4.65)$$

Since $y_m^k\in\mathrm{Ker}\triangle^*(\overline{\lambda_m^k})$, then for any $x\in\mathbb{C}^n$ we have:

$$0=\langle x,\triangle^*(\overline{\lambda_m^k})y_m^k\rangle=\langle\triangle(\lambda_m^k)x,y_m^k\rangle.$$

Therefore, for any θ the relation $\langle e^{-\lambda_m^k\theta}\triangle(\lambda_m^k)\xi(\theta),y_m^k\rangle=0$ holds, and, integrating it by θ from -1 to 0, we obtain:

$$0=\langle\int_{-1}^{0}e^{-\lambda_m^k\theta}\triangle(\lambda_m^k)\xi(\theta)\ \mathrm{d}\theta,y_m^k\rangle=\langle-\int_{-1}^{0}\lambda_m^k e^{-\lambda_m^k\theta}\xi(\theta)\ \mathrm{d}\theta+\int_{-1}^{0}\lambda_m^k e^{-\lambda_m^k\theta}A_{-1}\xi(\theta)\ \mathrm{d}\theta$$

$$+\lambda_m^k\int_{-1}^{0}e^{\lambda_m^k s}A_2(s)\ \mathrm{d}s\int_{-1}^{0}e^{-\lambda_m^k\theta}\xi(\theta)\ \mathrm{d}\theta+\int_{-1}^{0}e^{\lambda_m^k s}A_3(s)\ \mathrm{d}s\int_{-1}^{0}e^{-\lambda_m^k\theta}\xi(\theta)\ \mathrm{d}\theta,y_m^k\rangle.\qquad(4.66)$$

Adding the relation (4.66) to (4.65) we see that the term $\int_{-1}^{0}\lambda_m^k e^{-\lambda_m^k\theta}\xi(\theta)\ \mathrm{d}\theta$ is reducing and in the last two terms we have $-\int_{-1}^{0}(\int_{-1}^{s}\ \mathrm{d}\theta)\ \mathrm{d}s+\int_{-1}^{0}(\int_{-1}^{0}\ \mathrm{d}\theta)\ \mathrm{d}s=\int_{-1}^{0}(\int_{s}^{0}\ \mathrm{d}\theta)\ \mathrm{d}s= -\int_{-1}^{0}(\int_{0}^{s}\ \mathrm{d}\theta)\ \mathrm{d}s$. Finally, making a change of the variables: $s=\theta$, $\theta=s$ in these two terms, we obtain:

$$0=\langle z+\lambda_m^k e^{-\lambda_m^k}A_{-1}\int_{-1}^{0}e^{-\lambda_m^k\theta}\xi(\theta)\ \mathrm{d}\theta-\int_{-1}^{0}A_2(\theta)\xi(\theta)\ \mathrm{d}\theta$$

$$-\int_{-1}^{0}e^{\lambda_m^k\theta}\left[\lambda_m^k A_2(\theta)+A_3(\theta)\right]\left[\int_0^\theta e^{-\lambda_m^k s}\xi(s)\ \mathrm{d}s\right]\ \mathrm{d}\theta,y_m^k\rangle$$

$$=\langle D(z,\xi,\lambda_m^k),y_m^k\rangle.$$

Since, $y_m^k\in\mathrm{Ker}\triangle^*(\overline{\lambda_m^k})$ we conclude that $D(z,\xi,\lambda_m^k)\perp\mathrm{Ker}\triangle^*(\overline{\lambda_m^k})$ and this is equivalent to the inclusion (4.62), i.e.

$$D(z,\xi,\lambda_m^k)\in\mathrm{Im}\triangle(\lambda_m^k).$$

In other words, there exists the inverse image of the vector $D(z,\xi,\lambda_m^k)$ though, and we emphasize this fact, the matrix $\triangle^{-1}(\lambda_m^k)$ does not exists ($\det\triangle(\lambda_m^k)=0$).

The last completes the proof of the lemma.

Proof of Lemma 4.2. We analyze the behavior of the vector-function $\triangle^{-1}(\lambda)D(z,\xi,\lambda)$ near the imaginary axis. The matrix $\triangle^{-1}(\lambda)$ does not exists for the values $\lambda=\lambda_m^k$, where

$\lambda_m^k \in \Lambda_1$ are eigenvalues of the operator $\mathcal{A}$ and these eigenvalues approach to the imaginary axis when $k \to \infty$. On the other hand, and we prove this fact below, the limit $\lim\limits_{\lambda \to \lambda_m^k} \triangle^{-1}(\lambda) D(z, \xi, \lambda)$ exists.

We introduce a notation:

$$f(\lambda) = \triangle^{-1}(\lambda) D(z, \xi, \lambda) = \left(\frac{1}{\lambda}\triangle(\lambda)\right)^{-1} \left(\frac{1}{\lambda} D(z, \xi, \lambda)\right). \tag{4.67}$$

1. In the first part of our proof we find a transformation that "separates" the singularity of the matrix $\left(\frac{1}{\lambda}\triangle(\lambda)\right)^{-1}$. Using this representation we will show that in a neighborhood $U(\lambda_m^k)$ of each eigenvalues $\lambda_m^k \in \Lambda_1$ the vector-function $f(\lambda)$ is analytic, i.e. we show that in the representation

$$f(\lambda) = \frac{1}{\lambda - \lambda_m^k} f_{-1} + f_0 + (\lambda - \lambda_m^k) f_1 + ...$$

the coefficient $f_{-1} = \lim\limits_{\lambda \to \lambda_m^k} (\lambda - \lambda_m^k) f(\lambda)$ is equal to zero. (We note that in the case when λ_m^k would be a pole of order more than 1 we would obtain $\lim\limits_{\lambda \to \lambda_m^k} (\lambda - \lambda_m^k) f(\lambda) = \infty$.)

According to Lemma 3.2 there exist matrices $P_{m,k}$, $Q_{m,k}$ such that the matrix-function $\widehat{\triangle}_{m,k}(\lambda) \stackrel{\text{def}}{=} \frac{1}{\lambda} P_{m,k} R_m \triangle(\lambda) R_m Q_{m,k}$ at the point $\lambda = \lambda_m^k$ has the form (3.27), i.e.

$$\widehat{\triangle}_{m,k}(\lambda_m^k) \stackrel{\text{def}}{=} \frac{1}{\lambda_m^k} P_{m,k} R_m \triangle(\lambda_m^k) R_m Q_{m,k} = \begin{pmatrix} 0 & 0 \;\ldots\; 0 \\ 0 & \\ \vdots & S_{m,k} \\ 0 & \end{pmatrix}, \quad \det S_{m,k} \neq 0,$$

We rewrite the representation (4.67) of the function $f(\lambda)$ in a neighborhood $U(\lambda_m^k)$ as follows:

$$\begin{aligned} f(\lambda) &= \left(\frac{1}{\lambda} R_m P_{m,k}^{-1} P_{m,k} R_m \triangle(\lambda) R_m Q_{m,k} Q_{m,k}^{-1} R_m\right)^{-1} \left(\frac{1}{\lambda} D(z, \xi, \lambda)\right) \\ &= R_m Q_{m,k} \left(\frac{1}{\lambda} P_{m,k} R_m \triangle(\lambda) R_m Q_{m,k}\right)^{-1} P_{m,k} R_m \left(\frac{1}{\lambda} D(z, \xi, \lambda)\right) \\ &= R_m Q_{m,k} \left(\widehat{\triangle}_{m,k}(\lambda)\right)^{-1} P_{m,k} R_m \left(\frac{1}{\lambda} D(z, \xi, \lambda)\right). \end{aligned} \tag{4.68}$$

The matrix-function $\widehat{\triangle}_{m,k}(\lambda)$ is analytic a neighborhood of the λ_m^k and we write down its Taylor expansion:

$$\widehat{\triangle}_{m,k}(\lambda) = \widehat{\triangle}_{m,k}(\lambda_m^k) + (\lambda - \lambda_m^k) \widehat{\triangle}'_{m,k}(\lambda_m^k) + \frac{1}{2}(\lambda - \lambda_m^k)^2 \widehat{\triangle}''_{m,k}(\lambda_m^k) + \ldots$$

On the other hand, due to Corollary (3.1), the matrix-function $\widehat{\triangle}_{m,k}(\lambda)$ allows the representation (3.29), i.e.

$$\widehat{\triangle}_{m,k}(\lambda) = \begin{pmatrix} (\lambda - \lambda_m^k) r_{11}(\lambda) & (\lambda - \lambda_m^k) r_{12}(\lambda) \;\ldots\; (\lambda - \lambda_m^k) r_{1n}(\lambda) \\ (\lambda - \lambda_m^k) r_{21}(\lambda) & \\ \vdots & M_{m,k}(\lambda) \\ (\lambda - \lambda_m^k) r_{n1}(\lambda) & \end{pmatrix}, \quad \lambda \in U(\lambda_m^k),$$

where $r_{ij}(\lambda) = r_{ij}^{m,k}(\lambda)$ are analytic functions, and differentiating this representation by λ at the point $\lambda = \lambda_m^k$, we obtain:

$$\widehat{\triangle}'_{m,k}(\lambda_m^k) = \begin{pmatrix} r_{11}(\lambda_m^k) & r_{12}(\lambda_m^k) & \dots & r_{1n}(\lambda_m^k) \\ r_{21}(\lambda_m^k) & & & \\ \vdots & & M'_{m,k}(\lambda_m^k) & \\ r_{n1}(\lambda_m^k) & & & \end{pmatrix}$$

$$= \begin{pmatrix} r_{11}(\lambda_m^k) & r_{12}(\lambda_m^k) & \dots & r_{1n}(\lambda_m^k) \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix} + \begin{pmatrix} 0 & 0 & \dots & 0 \\ r_{21}(\lambda_m^k) & & & \\ \vdots & & M'_{m,k}(\lambda_m^k) & \\ r_{n1}(\lambda_m^k) & & & \end{pmatrix} = \Gamma_0 + \Gamma_1.$$

We introduce the matrix $F(\lambda) = \widehat{\triangle}_{m,k}(\lambda_m^k) + (\lambda - \lambda_m^k)\Gamma_0$, i.e.

$$F(\lambda) = \begin{pmatrix} r_{11}(\lambda_m^k)(\lambda - \lambda_m^k) & r_{12}(\lambda_m^k)(\lambda - \lambda_m^k) & \dots & r_{1n}(\lambda_m^k)(\lambda - \lambda_m^k) \\ 0 & & & \\ \vdots & & S_{m,k} & \\ 0 & & & \end{pmatrix}. \tag{4.69}$$

The matrix $F(\lambda)$ is non-singular in a neighborhood $U(\lambda_m^k)$. Indeed, $\det F(\lambda) = r_{11}(\lambda_m^k)(\lambda - \lambda_m^k)\det S_{m,k}$, where $\det S_{m,k} \neq 0$ and $r_{11}(\lambda_m^k) = 0$ due to Corollary (3.1).

Therefore, there exists the matrix $F^{-1}(\lambda)$, which has the following structure:

$$F^{-1}(\lambda) = \frac{1}{r_{11}(\lambda_m^k)(\lambda - \lambda_m^k)\det S_{m,k}} \begin{pmatrix} \det S_{m,k} & F_{21} & \dots & F_{n1} \\ 0 & F_{22} & \dots & F_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & F_{2n} & \dots & F_{nn} \end{pmatrix}, \tag{4.70}$$

where main minors F_{ij} of the matrix $F(\lambda)$ are of the form $F_{ij} = (\lambda - \lambda_m^k)f_{ij}$, $f_{ij} \in \mathbb{C}$.

Thus, we have the following representation of the matrix $\widehat{\triangle}_{m,k}(\lambda)$:

$$\widehat{\triangle}_{m,k}(\lambda) = F(\lambda) + (\lambda - \lambda_m^k)\Gamma_1 + \frac{1}{2}(\lambda - \lambda_m^k)^2\widehat{\triangle}''_{m,k}(\lambda_m^k) + \dots$$

$$= F(\lambda)(I + (\lambda - \lambda_m^k)F^{-1}(\lambda)\Gamma_1 + \frac{1}{2}(\lambda - \lambda_m^k)^2 F^{-1}(\lambda)\widehat{\triangle}''_{m,k}(\lambda_m^k) + \dots) \tag{4.71}$$

We introduce the notation $\Upsilon(\lambda) = (\lambda-\lambda_m^k)F^{-1}(\lambda)\Gamma_1 + \frac{1}{2}(\lambda-\lambda_m^k)^2 F^{-1}(\lambda)\widehat{\triangle}''_{m,k}(\lambda_m^k) + \dots$. Our next goal is to prove that there exists $\delta > 0$ such that in any neighborhood the following estimate holds:

$$\|\Upsilon(\lambda)\| \leq 1, \qquad \lambda \in U_\delta(\widetilde{\lambda}_m^k). \tag{4.72}$$

From (4.71) we have that $\Upsilon(\lambda) = F_{m,k}^{-1}(\lambda)\widehat{\triangle}_{m,k}(\lambda) - I$, i.e. we want to prove that $\forall\varepsilon > 0$, $\exists\delta > 0$, $\exists N$ such that $\forall|k| > N$, $\forall\lambda \in U_\delta(\widetilde{\lambda}_m^k)$ we have an estimate

$$\|F_{m,k}^{-1}(\lambda)\widehat{\triangle}_{m,k}(\lambda) - I\| \leq 1. \tag{4.73}$$

Let us introduce a notation for components of the matrices $\Upsilon(\lambda)$ and $M_{m,k}(\lambda)$: $\Upsilon(\lambda) = \{\Upsilon_{ij}(\lambda)\}_{i,j=1}^n$, $M_{m,k}(\lambda) = \{m_{ij}(\lambda)\}_{i,j=2}^n$.

Direct computations gives us that

$$F_{i1} = (-1)^{j+1}(\lambda - \lambda_m^k) \begin{vmatrix} r_{12}(\lambda_m^k) & \dots & r_{1n}(\lambda_m^k) \\ m_{22}(\lambda_m^k) & \dots & m_{2n}(\lambda_m^k) \\ \vdots & \vdots & \vdots \\ m_{i-12}(\lambda_m^k) & \dots & m_{i-1n}(\lambda_m^k) \\ m_{i+12}(\lambda_m^k) & \dots & m_{i+1n}(\lambda_m^k) \\ \vdots & \vdots & \vdots \\ m_{n2}(\lambda_m^k) & \dots & m_{nn}(\lambda_m^k) \end{vmatrix}, \qquad i = \overline{2, n},$$

$$F_{ij} = r_{11}(\lambda_m^k)(\lambda - \lambda_m^k) M_{ij}(\lambda_m^k), \qquad i, j = \overline{2, n},$$

where $M_{ij}(\lambda)$ are main minors of the matrix $M_{m,k}(\lambda)$. We conclude that F_{i1} converges to zero when $k \to \infty$. Let us compute the elements of the matrix $\Upsilon(\lambda) = F_{m,k}^{-1}(\lambda)\widehat{\triangle}_{m,k}(\lambda) - I$.

$$\Upsilon_{11}(\lambda) = \frac{(\lambda - \lambda_m^k) r_{11}(\lambda) \det M_{m,k}(\lambda_m^k) + (\lambda - \lambda_m^k) \sum_{j=2}^n r_{j1}(\lambda) F_{j1}}{(\lambda - \lambda_m^k) r_{11}(\lambda_m^k) \det M_{m,k}(\lambda_m^k)} - 1$$

$$= \frac{r_{11}(\lambda)}{r_{11}(\lambda_m^k)} + \frac{\sum_{j=2}^n r_{j1}(\lambda) F_{j1}}{r_{11}(\lambda_m^k) \det M_{m,k}(\lambda_m^k)} - 1 \to 0, \quad k \to \infty.$$

$$\Upsilon_{ii}(\lambda) = \frac{\sum_{j=2}^n m_{ji}(\lambda) F_{ji}}{(\lambda - \lambda_m^k) r_{11}(\lambda_m^k) \det M_{m,k}(\lambda_m^k)} - 1 = \frac{\sum_{j=2}^n m_{ji}(\lambda) M_{ji}(\lambda_m^k)}{\det M_{m,k}(\lambda_m^k)} - 1$$

$$= \frac{\sum_{j=2}^n (m_{ji}(\lambda) - m_{ji}(\lambda_m^k)) M_{ji}(\lambda_m^k)}{\det M_{m,k}(\lambda_m^k)} \to 0, \quad k \to \infty, \qquad i = \overline{2, n}.$$

$$\Upsilon_{1i}(\lambda) = \frac{(\lambda - \lambda_m^k) r_{1i}(\lambda) \det M_{m,k}(\lambda_m^k) + \sum_{j=2}^n m_{ji}(\lambda) F_{j1}}{(\lambda - \lambda_m^k) r_{11}(\lambda_m^k) \det M_{m,k}(\lambda_m^k)}$$

$$= \frac{r_{1i}(\lambda)}{r_{11}(\lambda_m^k)} + \frac{\sum_{j=2}^n m_{ji}(\lambda) F_{j1}}{(\lambda - \lambda_m^k) r_{11}(\lambda_m^k) \det M_{m,k}(\lambda_m^k)} \to 0, \quad k \to \infty, \qquad i = \overline{2, n}.$$

$$\Upsilon_{ij}(\lambda) = \frac{\sum_{s=2}^n m_{sj}(\lambda) F_{sj}}{(\lambda - \lambda_m^k) r_{11}(\lambda_m^k) \det M_{m,k}(\lambda_m^k)} = \frac{\sum_{s=2}^n m_{sj}(\lambda) M_{sj}(\lambda_m^k)}{\det M_{m,k}(\lambda_m^k)}$$

$$= \frac{\sum_{s=2}^n (m_{sj}(\lambda) - m_{sj}(\lambda_m^k)) M_{sj}(\lambda_m^k)}{\det M_{m,k}(\lambda_m^k)} \to 0, \quad k \to \infty, \qquad i, j = \overline{2, n}, i \neq j.$$

Therefore, the matrix $I + \Upsilon(\lambda)$ has an inverse for any $\lambda \in U_\delta(\widetilde{\lambda}_m^k)$:

$$(I + \Upsilon(\lambda))^{-1} = I + (\lambda - \lambda_m^k)\Gamma(\lambda), \tag{4.74}$$

where $\Gamma(\lambda)$ is analytic in a neighborhood $U(\lambda_m^k)$.

We note also that the matrix $F^{-1}(\lambda)\Gamma_1$ does not depend on λ. Indeed, denoting by $\widetilde{f}_{ij} = \frac{f_{ij}}{r_{11}(\lambda_m^k) \det S_{m,k}}$ we obtain:

$$F^{-1}(\lambda)\Gamma_1 = \begin{pmatrix} \frac{1}{r_{11}(\lambda_m^k)(\lambda - \lambda_m^k)} & \widetilde{f}_{21} & \dots & \widetilde{f}_{n1} \\ 0 & \widetilde{f}_{22} & \dots & \widetilde{f}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \widetilde{f}_{2n} & \dots & \widetilde{f}_{nn} \end{pmatrix} \begin{pmatrix} 0 & 0 \ \dots \ 0 \\ r_{21}(\lambda_m^k) & \\ \vdots & M'_{m,k}(\lambda_m^k) \\ r_{n1}(\lambda_m^k) & \end{pmatrix} = \widetilde{\Gamma}_1.$$

Therefore, the term $(\lambda-\lambda_m^k)F^{-1}(\lambda)\Gamma_1=(\lambda-\lambda_m^k)\widetilde{\Gamma}_1$ is small with respect to $(\lambda-\lambda_m^k)$ when $\lambda\to\lambda_m^k$. It is also easy to see that for any $s\geq 2$: $(\lambda-\lambda_m^k)F^{-1}(\lambda)\widehat{\triangle}_{m,k}^{(s)}(\lambda_m^k)=(\lambda-\lambda_m^k)\widetilde{\Gamma}_s(\lambda)$, where $\widetilde{\Gamma}_s(\lambda)$ are analytic matrices.

Finally, from (4.68), (4.71) and (4.74) we obtain:

$$f(\lambda)=R_mQ_{m,k}\widehat{\triangle}_{m,k}^{-1}(\lambda)P_{m,k}R_m\left(\frac{1}{\lambda}D(z,\xi,\lambda)\right)$$

$$=R_mQ_{m,k}\left(F(\lambda)(I+(\lambda-\lambda_m^k)F^{-1}(\lambda)\Gamma_1+\frac{1}{2}(\lambda-\lambda_m^k)^2F^{-1}(\lambda)\widehat{\triangle}''_{m,k}(\lambda_m^k)+\dots)\right)^{-1}$$

$$\cdot R_mP_{m,k}\left(\frac{1}{\lambda}D(z,\xi,\lambda)\right)=R_mQ_{m,k}(I+(\lambda-\lambda_m^k)\Gamma(\lambda))F^{-1}(\lambda)P_{m,k}R_m\left(\frac{1}{\lambda}D(z,\xi,\lambda)\right). \quad (4.75)$$

Since, due to Lemma (4.1), $D(z,\xi,\lambda_m^k)\in \mathrm{Im}\triangle(\lambda_m^k)$ (and, obviously $\frac{1}{\lambda_m^k}D(z,\xi,\lambda_m^k)\in \mathrm{Im}(\frac{1}{\lambda_m^k}\triangle(\lambda_m^k))$), then, due to Proposition 4.1 we have that

$$\frac{1}{\lambda_m^k}P_{m,k}R_mD(z,\xi,\lambda_m^k)\in \mathrm{Im}\widehat{\triangle}(\lambda_m^k). \quad (4.76)$$

Moreover, since the matrix $\widehat{\triangle}(\lambda_m^k)$ is of the form (3.27) we conclude that

$$\frac{1}{\lambda_m^k}P_{m,k}R_mD(z,\xi,\lambda_m^k)=(0,d_2^0,\dots,d_n^0)^T. \quad (4.77)$$

Since $D(z,\xi,\lambda)$ is an analytic vector-function in a neighborhood $U(\lambda_m^k)$ then $\frac{1}{\lambda}P_{m,k}R_mD(z,\xi,\lambda)$ is also an analytic vector-function and, due to (4.77) we conclude that

$$\frac{1}{\lambda}P_{m,k}R_mD(z,\xi,\lambda)=\begin{pmatrix}(\lambda-\lambda_m^k)d_1^1+\dots\\ d_2^0+(\lambda-\lambda_m^k)d_2^1+\dots\\ \vdots\\ d_n^0+(\lambda-\lambda_m^k)d_2^1+\dots\end{pmatrix}. \quad (4.78)$$

Using (4.75), (4.70) and (4.78), we obtain that

$$\lim_{\lambda\to\lambda_m^k}(\lambda-\lambda_m^k)f(\lambda)=R_mQ_{m,k}\lim_{\lambda\to\lambda_m^k}(\lambda-\lambda_m^k)F^{-1}(\lambda)P_{m,k}R_m\left(\frac{1}{\lambda}D(z,\xi,\lambda)\right)$$

$$=R_mQ_{m,k}\lim_{\lambda\to\lambda_m^k}\begin{pmatrix}\frac{1}{r_{11}(\lambda_m^k)} & (\lambda-\lambda_m^k)\widetilde{f}_{21} & \dots & (\lambda-\lambda_m^k)\widetilde{f}_{n1}\\ 0 & (\lambda-\lambda_m^k)\widetilde{f}_{22} & \dots & (\lambda-\lambda_m^k)\widetilde{f}_{n2}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & (\lambda-\lambda_m^k)\widetilde{f}_{2n} & \dots & (\lambda-\lambda_m^k)\widetilde{f}_{nn}\end{pmatrix}\begin{pmatrix}(\lambda-\lambda_m^k)d_1^1+\dots\\ d_2^0+\dots\\ \vdots\\ d_n^0+\dots\end{pmatrix}$$

$$=R_mQ_{m,k}\begin{pmatrix}\frac{1}{r_{11}(\lambda_m^k)} & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0\end{pmatrix}\begin{pmatrix}0\\ d_2^0\\ \vdots\\ d_n^0\end{pmatrix}=R_mQ_{m,k}\cdot 0=0. \quad (4.79)$$

Thus, we have proved that $f(\lambda)=\triangle^{-1}(\lambda)D(z,\xi,\lambda)$ is an analytic vector-function in a neighborhood of each eigenvalue λ_m^k.

2. Let us prove that $f(\lambda)$ is uniformly bounded in the neighborhoods of the eigenvalues $\lambda_m^k \in \Lambda_1$. This means to prove that the set of vectors

$$f_0 = f_0^{m,k} = f(\lambda_m^k) = (\triangle^{-1}(\lambda) D(z,\xi,\lambda))_{\lambda=\lambda_m^k}$$

is bounded. Taking into account the representation (4.75) we have:

$$f_0^{m,k} = f(\lambda_m^k) = \left(R_m Q_{m,k} F^{-1}(\lambda) P_{m,k} R_m \left(\frac{1}{\lambda} D(z,\xi,\lambda) \right) \right)_{\lambda=\lambda_m^k}$$

$$= R_m Q_{m,k} \begin{pmatrix} \frac{1}{(\lambda-\lambda_m^k) r_{11}(\lambda_m^k)} & \widetilde{f}_{21} & \dots & \widetilde{f}_{n1} \\ 0 & \widetilde{f}_{22} & \dots & \widetilde{f}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \widetilde{f}_{2n} & \dots & \widetilde{f}_{nn} \end{pmatrix} \begin{pmatrix} (\lambda-\lambda_m^k) d_1^1 + \dots \\ d_2^0 + \dots \\ \vdots \\ d_n^0 + \dots \end{pmatrix}_{\lambda=\lambda_m^k}$$

$$= R_m Q_{m,k} \begin{pmatrix} \frac{d_1^1}{r_{11}(\lambda_m^k)} + \sum\limits_{i=2}^{n} \widetilde{f}_{i1} d_i^0 \\ \sum\limits_{i=2}^{n} \widetilde{f}_{i2} d_i^0 \\ \vdots \\ \sum\limits_{i=2}^{n} \widetilde{f}_{in} d_i^0 \end{pmatrix}. \tag{4.80}$$

We recall that $\widetilde{f}_{ij} = \frac{f_{ij}}{r_{11}(\lambda_m^k) \det S_{m,k}}$ and, therefore, it remains to give estimates of the following values: $\|Q_{m,k}\|$, $\|P_{m,k}\|$, $\det S_{m,k}$, $r_{11}(\lambda_m^k)$, f_{ij}, d_i^0 and d_1^1.

An estimate on $\|P_{m,k}\|$ and $\|Q_{m,k}\|$ follows from Lemma 3.3 and, namely, from the estimate (3.28): for any $\varepsilon > 0$ there exists $N \in \mathbb{Z}$ such that for any $|k| \ge N$: $\|Q_{m,k}\| \le \sqrt{n + (n-1)\varepsilon^2} = C$, $\|P_{m,k}\| \le \sqrt{n+(n-1)\varepsilon^2} = C$.

An estimate $0 < C_2 \le |r_{11}(\lambda_m^k)|$ follows from the relation (3.30) of the Corollary 3.1, and thus

$$\frac{1}{|r_{11}(\lambda_m^k)|} \le \frac{1}{C_2}.$$

Estimates for $\det S_k$, f_{ij}, d_i^0 and d_1^1 follows immediately from Lemma (4.3).

Thus, we conclude that $\|f_0^{m,k}\| \le C$ for all $1 \le m \le \ell_1$, $k : |k| \ge N$ what completes the proof of the Lemma. □

5 Proof of auxiliary results

Proof of Proposition 4.1. The relation $y \in \mathrm{Im} A$ means that there exists a vector x such that $Ax = y$. Since Q is non-singular then there exists a vector x_1 such that $x = Qx_1$. Therefore, $AQx_1 = y$ and, multiplying on P from the left we obtain $PAQx_1 = Py$ what completes the proof of the proposition. □

Proof of Proposition 4.2. The proposition holds since the determinant is a continuous function of matrix elements. More precisely, since $\det(A) = \sum\limits_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod\limits_{i=1}^{n} a_{i,\sigma(i)}$, where S_n denotes the set of all $n!$ permutations of the set $S = \{1, 2, \dots, n\}$ we have

$$\det(A+B) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} (a_{i,\sigma(i)} + b_{i,\sigma(i)}) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \sum_{c_{i,\sigma(i)} = a_{i,\sigma(i)} \vee b_{i,\sigma(i)}} \prod_{i=1}^{n} c_{i,\sigma(i)}$$

$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)} + \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \sum_{condition} \prod_{i=1}^n c_{i,\sigma(i)}$$

$$= \det A + \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \sum_{condition} \prod_{i=1}^n c_{i,\sigma(i)},$$

where *condition* $\equiv (c_{i,\sigma(i)} = a_{i,\sigma(i)} \vee b_{i,\sigma(i)}$ and there is always at least one multiplier $b_{i,\sigma(i)})$. (The last notation is not too good, but the proposition is just for internal utilization :)).

The sum $\sum\limits_{\sigma\in S_n}$ consists of $n!$ terms and the sum $\sum\limits_{c_{i,\sigma(i)}=a_{i,\sigma(i)}\vee b_{i,\sigma(i)}}$ consists of 2^n terms. Therefore, the sum $\sum\limits_{\sigma\in S_n}\sum\limits_{condition}$ consists of $n!(2^n-1)$ terms. Thus, choosing $\varepsilon_0 < \frac{|\det A|}{2C}$, where $C = n!(2^n-1)(\max\limits_{i,j}|a_{ij}|)^{n-1}$, we obtain

$$|\det(A+B)| \geq \left| |\det A| - |\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \sum_{condition} \prod_{i=1}^n c_{i,\sigma(i)}| \right| \geq |\det A| - \frac{1}{2}|\det A| = \frac{1}{2}|\det A| > 0$$

what completes the proof of the proposition. $\square$

Proof of Proposition 4.3. Integrals $a_k(\lambda)$ can be considered as Fourier coefficients of the function $e^{\lambda s}f(s)$, thus, they converge to zero when $k \to \infty$. It remains to prove that they converge uniformly on the set L_0. The last means that for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for any $|k| \geq n$ and for any $\lambda \in L_0$ we have $|a_k(\lambda)| < \varepsilon$.

Let us suppose the contrary: $\exists\varepsilon > 0$ such that $\forall n \in \mathbb{N}$, $\exists |k| \geq n$, $\exists \lambda \in L_0$: $|a_k(\lambda)| \geq \varepsilon$. Thus, there exists a sequence $k_1 < k_2 < \dots$ and a sequence $\{\lambda_{k_i}\}_{i=1}^\infty$ such that $|a_{k_i}(\lambda_{k_i})| \geq \varepsilon$.

Since L_0 is a bounded set then there exists a converging subsequence of $\{\lambda_{k_i}\}_{i=1}^\infty$ which we denote by $\{\lambda_j\}_{j\in J}$, where $J \subset N$ is a strictly increasing sequence. Moreover, since L_0 is also closed, then the limit of $\{\lambda_j\}_{j\in J}$ belongs to L_0: $\lambda_j \to \lambda_0 \in L_0$. Let us show that the sequence $\{a_k(\lambda_0)\}$ does not converge to zero.

Indeed, choosing big enough $n \in \mathbb{N}$, such that for any $j > n$, $j \in J$ and any $s \in [-1, 0]$: $|e^{\lambda_0 s} - e^{\lambda_j s}| \leq \varepsilon/2\|f(s)\|$, we obtain

$$|a_j(\lambda_0) - a_j(\lambda_j)| = |\int_{-1}^0 e^{2\pi i j s}(e^{\lambda_0 s} - e^{\lambda_j s})f(s)\,\mathrm{d}s| \leq \int_{-1}^0 |e^{\lambda_0 s} - e^{\lambda_j s}|f(s)\,\mathrm{d}s \leq \frac{\varepsilon}{2}.$$

Since $|a_j(\lambda_j)| \geq \varepsilon$ and assuming that $|a_j(\lambda_0)| \leq |a_j(\lambda_j)|$, we obtain

$$\frac{\varepsilon}{2} \geq |a_j(\lambda_0) - a_j(\lambda_j)| \geq |a_j(\lambda_j)| - |a_j(\lambda_0)| \geq \varepsilon - |a_j(\lambda_0)|,$$

and, thus, $|a_j(\lambda_0)| \geq \frac{\varepsilon}{2}$ for any $j \in J$, $j > n$.

Thus, $\{a_k(\lambda_0)\}$ does not converge to zero and we have obtained a contradiction with the fact that they are the coefficients of the Fourier series of the function $e^{\lambda_0 s}f(s)$. The last completes the proof of the proposition. $\square$

6 An example of stable and unstable situations

In this section we give an example of systems having the same spectrum but some of them are stable and some are unstable. The spectrum of these systems satisfies the following assumptions:

$$\sigma(\mathcal{A}) = \{\lambda : \mathrm{Re}\lambda < 0\}, \quad \sigma(A_{-1}) = \{\mu : |\mu| \leq 1\} \Rightarrow \sigma(\widetilde{\mathcal{A}}) = \{\lambda : \mathrm{Re}\lambda \leq 0\},$$

and it satisfies the assumption that the set $\sigma_1 = \sigma(A_{-1}) \cap \{\mu : |\mu| = 1\}$ is not empty and no eigenvalue $\mu \in \sigma_1$ generates a Jordan block of dimension higher than one.

We consider the system

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + A_0 z(t), \tag{6.81}$$

with the state space $\mathbb{C}^2$. The matrices A_{-1} and A_0 are of the form:

$$A_{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -b & s \\ 0 & -b \end{pmatrix},$$

where b is a real positive number and for the value of s we essentially distinguish two cases: $s = 0$ and $s \neq 0$.

Remark 6.1 *The system (6.81) is a special case of the system (1.1). To show this one should choose $A_2(\theta) = (\theta+1)A_0$ and $A_3(\theta) = A_0$. Thus,*

$$\int_{-1}^{0} A_2(\theta)\dot{z}(t+\theta)\,\mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)z(t+\theta)\,\mathrm{d}\theta = \int_{-1}^{0} (\theta+1)A_0\dot{z}(t+\theta)\,\mathrm{d}\theta + \int_{-1}^{0} A_0 z(t+\theta)\,\mathrm{d}\theta =$$

$$= A_0 \int_{-1}^{0} ((\theta+1)z(t+\theta))'\,\mathrm{d}\theta = A_0 z(t).$$

We prove the following three propositions about systems (6.81).

Proposition 6.1 *For any $b > 0$ and any $s \in \mathbb{C}$ the spectrum of the corresponding operator $\mathcal{A}$ belongs to the open left half-plane, i.e. for any $\lambda \in \sigma(\mathcal{A})$: $\mathrm{Re}\lambda < 0$.*

Proposition 6.2 *If $s = 0$ then the operator $\mathcal{A}$ possesses eigenvectors only, i.e. it possesses no root vectors; if $s \neq 0$ then to any eigenvalue $\lambda \in \sigma(\mathcal{A})$ there corresponds a pair of eigenvector and root vector of the operator $\mathcal{A}$.*

Proposition 6.3 *If $s = 0$ then the system (6.81) is stable and, if $s \neq 0$ then it is unstable.*

Proof of Proposition 6.1. The eigenvalues of the operator $\mathcal{A}$ are determined from the equation

$$\det \triangle_{\mathcal{A}}(\lambda) = \det \left(-\lambda I + \lambda e^{-\lambda} A_{-1} + \lambda \int_{-1}^{0} e^{\lambda s} A_2(s)\,\mathrm{d}s + \int_{-1}^{0} e^{\lambda s} A_3(s)\,\mathrm{d}s \right) = 0,$$

which in our particular case has the form:

$$\det(-\lambda I + \lambda e^{-\lambda} A_{-1} + A_0) = \det \begin{pmatrix} -\lambda - \lambda e^{-\lambda} - b & s \\ 0 & -\lambda - \lambda e^{-\lambda} - b \end{pmatrix} = 0.$$

Thus, all the eigenvalues of the operator $\mathcal{A}$ satisfy the equation

$$\lambda e^{\lambda} + \lambda + b e^{\lambda} = 0 \tag{6.82}$$

and the multiplicity of each of these eigenvalues (as zeroes of the equation $\det \triangle_{\mathcal{A}}(\lambda) = 0$) equals two. Further, we list some results on transcendental equations obtained in the paper of L. Pontryagin [17].

Let us consider an equation $H(z) = 0$, where $H(z) = h(z, e^z)$ is a polynomial with respect to z and e^z.

Definition 6.1 *We say that the function* $H(z) = \sum\limits_{m,n} a_{mn} z^m e^{nz}$ *possesses the principal term* $a_{rs} z^r e^{sz}$*, if for all other terms* $a_{mn} z^m e^{nz}$ *we have that* $r \geq m$ *and* $s \geq n$*.*

We denote by $F(y)$ and $G(y)$ the real-valued functions of a real variable which are correspondingly the real and the imaginary parts of the function $H(iy)$, i.e. $H(iy) = F(y) + iG(y)$.

Definition 6.2 *We say that the zeroes of two real-valued functions of a real variable alternate iff*

a) each of these functions has no multiple roots;

b) between every two zeroes of one of these functions there exists at least one zero of the other;

c) the functions are never simultaneously zero.

Theorem 6.1 ([17, Pontryagin, 1955]). *Let* $H(z) = h(z, e^z)$ *be a polynomial with a principal term.*

1. If all the zeroes of the function $H(z)$ *lie to the left side of the imaginary axis (*$\mathrm{Re}\lambda_k < 0$*), then the zeroes of the functions* $F(y)$ *and* $G(y)$ *are real, alternating and for all* $y \in \mathbb{R}$ *the following inequality holds*

$$G'(y)F(y) - G(y)F'(y) > 0. \tag{6.83}$$

2. Any of the conditions below is sufficient for all the zeroes of the function $H(z)$ *to lie in the open left half-plane:*

a) all the zeroes of the functions $F(y)$ *and* $G(y)$ *are real, alternating and the inequality (6.83) is satisfied for at least one value of* y*;*

b) all the zeroes of the function $F(y)$ *are real and for each zero* $y = y_0$ *the inequality (6.83) is satisfied, i.e.* $G(y_0)F'(y_0) < 0$*;*

c) all the zeroes of the function $G(y)$ *are real and for each zero* $y = y_0$ *the inequality (6.83) is satisfied, i.e.* $G'(y_0)F(y_0) > 0$*.*

The following theorem allows us to check whether all zeroes of a function are real.

Theorem 6.2 ([17, Pontryagin, 1955]). *Let* $F(z) = f(z, \cos z, \sin z)$ *be a polynomial with a principal term* $z^r \phi_m^{(s)}(\cos z, \sin z)$*, where* $\phi_m^{(s)}(\cos z, \sin z)$ *is a homogeneous with respect to* $\cos z$ *and* $\sin z$ *polynomial.*

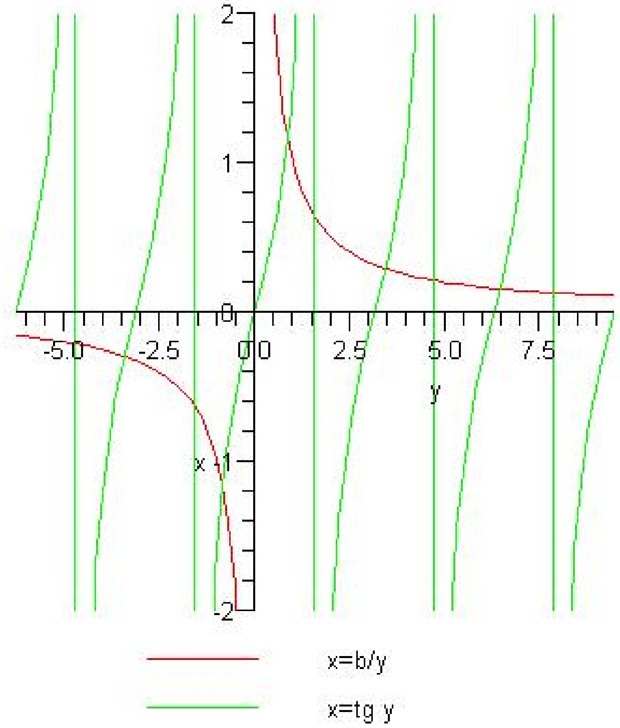
The function $F(z)$*,* $z \in \mathbb{C}$ *possesses only real zeroes if and only if for all big enough* $k \in \mathbb{Z}$ *the function* $F(x)$*,* $x \in \mathbb{R}$ *possesses exactly* $4ks + r$ *real roots on the interval* $-2\pi k + \varepsilon \leq x \leq 2\pi k + \varepsilon$ *for some* $\varepsilon > 0$*.*

We use the results mentioned above to analyze the equation (6.82). We divide $H(iy)$ onto real and imaginary parts:

$$H(iy) = iy(\cos y + i \sin y) + iy + b(\cos y + i \sin y)$$

$$= (b \cos y - y \sin y) + i(y \cos y + y + \sin y) = F(y) + iG(y).$$

and use the sufficient condition b) of Theorem 6.2. The equation $F(y) = 0$ can be rewritten in the form $\mathrm{tg} y = \frac{b}{y}$ (since all the zeroes of the equations $y = 0$ and $\cos y = 0$ are not zeroes of the equation $F(y) = 0$, we divide the initial equation on $y \cos y$). Some zeroes of the equation $F(y) = 0$ can be seen on the picture below:



From the picture we can see that exactly 5 zeroes of the equation $F(y) = 0$ belongs to the interval $-2\pi + \varepsilon \le y \le 2\pi + \varepsilon$ if we choose $\varepsilon > 0$ "big" enough such that the zero on the right from 2π belongs to the interval. Adding to this interval from the left and from the right 2π we add each time 4 other zeroes. Thus, on each interval $-2\pi k + \varepsilon \le y \le 2\pi k + \varepsilon$ we have exactly $4k+1$ zeroes of the equation $F(y) = 0$ and, therefore, the conditions of the Theorem 6.2 are satisfied. Thus, all the zeroes of the equation $F(y) = 0$ are real.

Let us now prove that for each root y_0 of the equation $F(y) = 0$ the inequality $G(y_0)F'(y_0) < 0$ holds. We use the notation $\cos y = C$, $\sin y = S$ and prove that $-G(y_0)F'(y_0) > 0$:

$$-GF' = -(-bS - S - y_0 C)(y_0 C + y_0 + bS) = (bS + S + y_0 C)(y_0 C + y_0 + bS).$$

We rewrite $\mathrm{tg} y_0 = \frac{b}{y_0}$ as $C = \frac{Sb}{y_0}$ and substitute C at the last relation:

$$-GF' = (bS + \frac{y_0^2}{b}S + S)(\frac{y_0^2}{b}S + bS + y_0) = S^2(b + \frac{y_0^2}{b} + 1)(\frac{y_0^2}{b} + b + \frac{y_0}{S}).$$

Since $S^2 > 0$ and $b + \frac{y_0^2}{b} + 1 > 0$, then it remains to prove, that the third multiplier is greater then zero. From $\mathrm{tg} y_0 = \frac{b}{y_0}$ we also have $b = \frac{y_0 S}{C}$ and substituting we obtain:

$$\frac{y_0^2}{b} + b + \frac{y_0}{S} = \frac{y_0 C}{S} + \frac{y_0 S}{C} + \frac{y_0}{S} = y_0 \frac{C^2 + S^2 + C}{SC} = y_0 \frac{1 + C}{SC} = (1 + C)bC^2 > 0.$$

We note also that the fact that $y_0 SC > 0$ can be easily seen from the picture above.

Applying Theorem 6.2 we complete the proof of the fact that all the eigenvalues of the operator $\mathcal{A}$ belongs to the open left half-plane. $\square$

Proof of Proposition 6.2. We consider two cases.

1. If $s = 0$ then $\triangle_{\mathcal{A}}(\lambda_k) \equiv 0$ for each zero λ_k of the equation (6.82), and, therefore, the the space of solutions of the equation $\triangle_{\mathcal{A}}(\lambda_k)z = 0$ is two-dimensional. We choose the following basis of this space: $z_k^1 = (1, 0)^T$, $z_k^2 = (0, 1)^T$.

Let us consider the equation for eigenvectors:

$$(\mathcal{A}-\lambda_k I)\left(\begin{array}{c} y \\ z(\theta) \end{array}\right)=0 \Leftrightarrow \left(\begin{array}{c} A_0 z(0)-\lambda_k y \\ \dot{z}(\theta)-\lambda_k z(\theta) \end{array}\right)=0.$$

The solution of the second equation is given by $z(\theta)=e^{\lambda_k\theta}z(0)$, thus, $z(-1)=e^{-\lambda_k}z(0)$. Taking into account the domain of the operator $\mathcal{A}$: $y=z(0)-A_{-1}z(-1)=(I-e^{-\lambda_k}A_{-1})z(0)$, we obtain from the first equation: $(A_0-\lambda_k I+\lambda_k e^{-\lambda_k}A_{-1})z(0)=0$, or, in an equivalent form: $\triangle_{\mathcal{A}}(\lambda_k)z(0)=0$.

As we have noted above the last equation has two-dimensional solution: $z(0)=z_k^1=(1,0)^T$ and $z(0)=z_k^2=(0,1)^T$, and therefore, there is a two-dimensional eigenspace of the operator $\mathcal{A}$ corresponding to the eigenvalue λ_k. Two eigenvectors of this subspace can be chosen in the following form: $f_k^1=\left(\begin{array}{c} y_k^1 \\ z_k^1(\cdot) \end{array}\right)$, where $y_k^1=\left(\begin{array}{c} 1+e^{-\lambda_k} \\ 0 \end{array}\right)$, $z_k^1(\theta)=\left(\begin{array}{c} e^{\lambda_k\theta} \\ 0 \end{array}\right)$ and $f_k^2=\left(\begin{array}{c} y_k^2 \\ z_k^2(\cdot) \end{array}\right)$, where $y_k^2=\left(\begin{array}{c} 0 \\ 1+e^{-\lambda_k} \end{array}\right)$, $z_k^2(\theta)=\left(\begin{array}{c} 0 \\ e^{\lambda_k\theta} \end{array}\right)$.

Thus, to any eigenvalue λ_k of the operator $\mathcal{A}$ there corresponds the two-dimensional eigenspace.

2. If $s=1$ (or $s\neq 0$) we have that $\triangle_{\mathcal{A}}(\lambda_k)=\left(\begin{array}{cc} 0 & e^{-\lambda_k} \\ 0 & 0 \end{array}\right)$ and, therefore, the space of solutions of the equation $\triangle_{\mathcal{A}}(\lambda_k)z=0$ is one-dimensional: $z_k^1=(1,0)^T$ (since, obviously, $e^{-\lambda_k}\neq 0$). Thus, the equation $\triangle_{\mathcal{A}}(\lambda_k)z=z_k^1$ has also one-dimensional solution which we denote by $z_k^2=(0,1)^T$.

Arguing as above we show that the operator $\mathcal{A}$ possesses one eigenvector corresponding to the eigenvalue λ_k: $f_k^1=\left(\begin{array}{c} y_k^1 \\ z_k^1(\cdot) \end{array}\right)$, where $y_k^1=\left(\begin{array}{c} 1+e^{-\lambda_k} \\ 0 \end{array}\right)$, $z_k^1(\theta)=\left(\begin{array}{c} e^{\lambda_k\theta} \\ 0 \end{array}\right)$.

Let us show that the operator $\mathcal{A}$ possesses one root vector. Each root vector f of the operator $\mathcal{A}$ satisfies the following equation: $(\mathcal{A}-\lambda_k I)^2 f=0$, i.e. $\widetilde{f}=(\mathcal{A}-\lambda_k I)f\in \mathrm{Ker}(\mathcal{A}-\lambda_k I)$. From the last we conclude that $\widetilde{f}=f_k^1=\left(\begin{array}{c} (1+e^{-\lambda_k},0)^T \\ (e^{\lambda_k\theta},0)^T \end{array}\right)$ and the root vector satisfies the relation

$$\left(\begin{array}{c} A_0 z(-1)-\lambda_k y \\ \dot{z}(\theta)-\lambda_k z(\theta) \end{array}\right)=\left(\begin{array}{c} (1+e^{-\lambda_k},0)^T \\ (e^{\lambda_k\theta},0)^T \end{array}\right).$$

The solution of the second equation is given by

$$z(\theta)=e^{\lambda_k\theta}z(0)+\int_0^\theta e^{\lambda_k(\theta-\tau)}e^{\lambda_k\tau}y_k^1\,\mathrm{d}\tau=e^{\lambda_k\theta}z(0)+\theta e^{\lambda_k\theta}y_k^1,$$

what implies $z(-1)=e^{-\lambda_k}z(0)-e^{-\lambda_k}y_k^1$. Taking into account the domain of the operator $\mathcal{A}$, we write down the first equation in the following form:

$$A_0 z(0)-\lambda_k(z(0)-A_{-1}(e^{-\lambda_k}z(0)-e^{-\lambda_k}y_k^1))=(1+e^{-\lambda_k})y_k^1$$

which is equivalent to

$$\triangle_{\mathcal{A}}(\lambda_k)z(0)=(1+e^{-\lambda_k})y_k^1+\lambda_k A_{-1}e^{-\lambda_k}y_k^1=\left(\begin{array}{c} 1+e^{-\lambda_k}-\lambda_k e^{-\lambda_k} \\ 0 \end{array}\right).$$

Thus, if $1+e^{-\lambda_k}-\lambda_k e^{-\lambda_k}\neq 0$ then $z(0)$ is the root vector of the matrix $\triangle_{\mathcal{A}}(\lambda_k)$ and, therefore, $z(0)=z_k^2=(0,1)^T$ and the operator $\mathcal{A}$ possesses the root vector $f_k^2=\begin{pmatrix} y_k^2 \\ z_k^2(\cdot)\end{pmatrix}$, where $y_k^2=\begin{pmatrix} e^{-\lambda_k} \\ 1+e^{-\lambda_k}\end{pmatrix}$ and $z_k^2(\theta)=\begin{pmatrix} \theta e^{\lambda_k\theta} \\ e^{\lambda_k\theta}\end{pmatrix}$.

It remains to show that $1+e^{-\lambda_k}-\lambda_k e^{-\lambda_k}\neq 0$. We suppose the contrary: $\lambda_k e^{-\lambda_k}=1+e^{-\lambda_k}$ and, multiplying the last expression onto e^{λ_k}, we obtain $\lambda_k=e^{\lambda_k}+1$. Since λ_k is an eigenvalue of $\mathcal{A}$, then we obtain $e^{\lambda_k}(e^{\lambda_k}+1)+e^{\lambda_k}+1+e^{\lambda_k}=0$ or $e^{2\lambda_k}+3e^{\lambda_k}+1=0$, and we conclude that $e^{\lambda_k}=\frac{-3\pm\sqrt{5}}{2}$. For the root $e^{\lambda_k}=\frac{-3-\sqrt{5}}{2}$ we have that $\mathrm{Re}\lambda_k>0$, and for the root $e^{\lambda_k}=\frac{-3+\sqrt{5}}{2}$ we have: $\lambda_k=\frac{-3+\sqrt{5}}{2}+1=\frac{\sqrt{5}-1}{2}>0$. We have obtained the contradiction what completes the analysis of eigenvectors and root vectors of the operator $\mathcal{A}$.

Proof of Proposition 6.3. Again we consider two cases.

1. First we consider the system (6.81) when $s=0$ and prove that in this case the system is stable. Let us evaluate the norm of eigenvectors.

$$\|f_k^1\|^2=\|f_k^2\|^2=\langle 1+e^{-\lambda_k},\overline{1+e^{-\lambda_k}}\rangle+\int_{-1}^{0}e^{\lambda_k\theta}\overline{e^{\lambda_k\theta}}\,\mathrm{d}\theta=|1+e^{-\lambda_k}|^2+\int_{-1}^{0}e^{2\mathrm{Re}\lambda_k\theta}\,\mathrm{d}\theta$$

$$=|1+e^{-\lambda_k}|^2+\frac{1}{2\mathrm{Re}\lambda_k}\left(1-e^{-2\mathrm{Re}\lambda_k}\right).$$

Since $\mathrm{Re}\lambda_k\to 0$ when $k\to\infty$, then $\lim\limits_{k\to\infty}\frac{1}{2\mathrm{Re}\lambda_k}\left(1-e^{-2\mathrm{Re}\lambda_k}\right)=1$. Therefore, $0<C_1\leq\frac{1}{2\mathrm{Re}\lambda_k}\left(1-e^{-2\mathrm{Re}\lambda_k}\right)\leq C_2$. Taking into account that $|e^{-\lambda_k}|\leq C_3$, we obtain the following estimates:

$$C_1\leq\|f_k^i\|^2\leq(1+C_3)^2+C_2=C_4. \tag{6.84}$$

As it has been shown in [18], the subspaces $V^{(k)}=\mathrm{Lin}\{f_k^1,f_k^2\}$ (and the finite-dimensional subspace W_N) form a Riesz basis of the space M_2. Since we have proved the estimate (6.84) then the eigenvectors $\{f_k^1,f_k^2\}$ (together with vectors from W_N) form a basis of M_2. Therefore, we have a Riesz basis of eigenvectors. Further we just repeat the proof given in [18, Theorem 23].

We consider a norm $\|\cdot\|_1$ in which the eigenvectors $\{f_k^1,f_k^2\}_{k\in\mathbb{Z}}$ are orthogonal. Let a vector x belongs to a closed span of the subspaces $V^{(k)}$, then $x=\sum\limits_{k\in\mathbb{Z}}(\alpha_k f_k^1+\beta_k f_k^2)$ and we have:

$$e^{\mathcal{A}t}x=\sum_{k\in\mathbb{Z}}e^{\lambda_k t}(\alpha_k f_k^1+\beta_k f_k^2).$$

Therefore,

$$\|e^{\mathcal{A}t}x\|_1^2=\sum_{k\in\mathbb{Z}}e^{\lambda_k t}(\|\alpha_k f_k^1\|_1^2+\|\beta_k f_k^2\|_1^2)\leq\sum_{k\in\mathbb{Z}}(\|\alpha_k f_k^1\|_1^2+\|\beta_k f_k^2\|_1^2)=\|x\|_1^2$$

and, thus, the family $e^{\mathcal{A}t}$ is uniformly bounded in the subspace generated by the subspaces $V^{(k)}$. From the last we conclude that the system is strongly asymptotically stable.

2. Let us consider the system (6.81) when $s=1$.

Let an operator $\mathcal{A}$ has a sequence of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ such that $\mathrm{Re}\lambda_k<0$ and $\mathrm{Re}\lambda_k\to 0$ when $k\to\infty$ and to each λ_k there corresponds one eigenvector v^k and at least one root vector w^k. We show that the equation $\dot{x}=\mathcal{A}x$ is unstable. Let us suppose that $\|v^k\|=\|w^k\|=1$. Since, for each w^k we have $e^{\mathcal{A}t}w^k=e^{\lambda_k t}(tv^k+w^k)$ then

$$\|e^{\mathcal{A}t}w^k\|=|e^{\lambda_k t}|\|(tv^k+w^k)\|\geq e^{\mathrm{Re}\lambda_k t}(t-1).$$

For any constant $C > 0$ we take $t \geq 2C + 1$ and for this t we take big enough k such that $e^{\mathrm{Re}\lambda_k t} \geq \frac{1}{2}$. Then we have:

$$\|e^{\mathcal{A}t} w^k\| \geq \frac{1}{2}(2C + 1 - 1) = C$$

and we conclude that $\|e^{\mathcal{A}t}\| \geq C$ for $t \geq 2C + 1$. Therefore, the family of exponents $e^{\mathcal{A}t}$ is not uniformly bounded and because of Banach-Steinhaus theorem there exists $x \in D(A)$ such that $\|e^{\mathcal{A}t} x\| \to \infty$ when $t \to +\infty$.

Thus, the system (6.81) is unstable when $s = 1$. The last completes the proof of the proposition. □

7 Conclusion

We have shown that the result on stability analysis obtained in [18] for neutral type systems can be extended on mixed retarded-neutral type systems. Though the formulation of the result remains the same, the method of its proof requires to involve resolvent boundedness technic.

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